Martingale defocusing and transience of a self-interacting random walk

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March 6, 2014

Abstract

Suppose that (X, Y, Z) is a random walk in $\mathbb{Z}^3$ that moves in the following way: on the first visit to a vertex only Z changes by ± 1 equally likely, while on later visits to the same vertex (X, Y) performs a two-dimensional random walk step. We show that this walk is transient thus answering a question of Benjamini, Kozma and Schapira. One important ingredient of the proof is a dispersion result for martingales.

Keywords and phrases. Transience, martingale, self-interacting random walk, excited random walk.
MSC 2010 *subject classifications.* Primary 60K35.

1   Introduction

In this paper we study the following self-interacting random walk (X, Y, Z) in $\mathbb{Z}^3$. On the first visit to a vertex only Z changes by ± 1 equally likely, while on later visits to the same vertex (X, Y) performs a two dimensional random walk step, i.e. it changes by $(\pm 1, 0)$ or $(0, \pm 1)$ all with equal probability. This walk was conjectured in [7] to be transient.

This model fits into the wider class of excited random walks which were first introduced by Benjamini and Wilson [8]. They study walks that on the first visit to a vertex in $\mathbb{Z}^d$ have a bias in one direction while on later visits they make a simple random walk step. There has been a lot of active research in this type of model; see the recent survey [11] and the references therein.

Another process of this flavour was analysed in [15]; suppose that μ_1, μ_2 are two zero-mean measures in $\mathbb{R}^3$ and consider any adapted rule for choosing between μ_1 and μ_2. By adapted rule, we mean that the next choice every time depends on the history of the process up to this time. In [15] it was proved that if each measure is supported on the whole space, then for any adapted rule, the resulting walk in $\mathbb{R}^3$ is transient. In [16] transience and recurrence properties and weak laws of large numbers were also proved for specific choices of one-dimensional measures; for instance when μ_1 is the distribution of simple random walk step and μ_2 the symmetric discrete Cauchy law.

A larger class of such processes are the so-called self-interacting random walks, which are not Markovian, since the next step depends on the whole history of the process up to the present time.

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For instance the edge or vertex reinforced random walks have attracted a lot of attention, see e.g. [2, 4, 9, 13, 14, 17, 18, 19].

Theorem 1.1. *Let $W_t = (X_t, Y_t, Z_t)$ be a random walk in $\mathbb{Z}^3$ such that on the first visit to a vertex only Z_t changes to $Z_t \pm 1$ equally likely, while on later visits to a vertex (X_t, Y_t) makes a two dimensional simple random walk step. Then W is transient, i.e. $\|W_t\| \to \infty$ as $t \to \infty$ almost surely.*
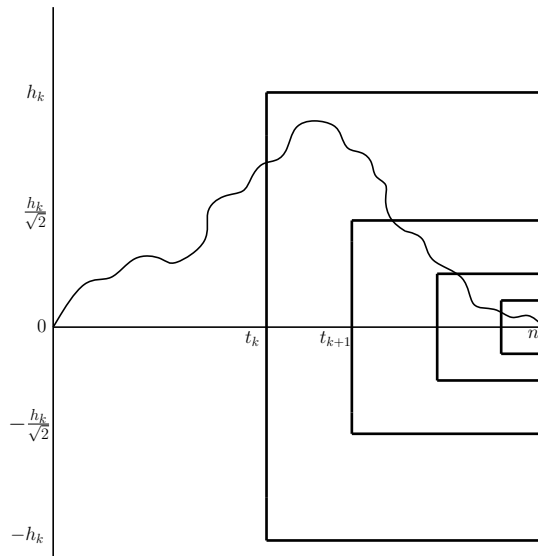

Figure 1: The rectangles $[t_k, n] \times [-h_k, h_k]$ and the graph of M

We now give a quick overview of the proof of Theorem 1.1. By conditioning on all the jumps of the two dimensional process (Y, Z) and looking at the process X only at the times when (Y, Z) moves, we obtain a martingale M. Then we need to obtain estimates for the probability that M is at 0 at time n so that when multiplied by $1/n$ it should be summable. In Section 2 we state and prove a proposition that gives estimates for a martingale to be at 0 at time n when it satisfies certain assumptions. We now state a simpler form of this proposition.

Corollary 1.2. *Let M be a martingale satisfying almost surely*

$$\mathbb{E}\big[(M_{k+1} - M_k)^2 \mid \mathcal{F}_k\big] \geq 1 \quad and \quad |M_{k+1} - M_k| \leq (\log n)^a,$$

for all $k \leq n$ and some $a < 1$. Then there exists a positive constant c, such that

$$\mathbb{P}(M_n = 0) \leq \exp\left(-c(\log n)^{1-a}\right).$$

We remark that related results were recently proved by Alexander in [1] and by Armstrong and Zeitouni in [3].

In order to prove Corollary 1.2 we use the same approach as in [10]. More specifically, we consider the rectangles as in Figure 1, where the widths decay exponentially and $t_k = n - n/2^k$ for $k < \log_2(n)$. It is clear that $\{M_n = 0\}$ only if the graph of M hits all the rectangles. Note that it suffices to show that for most rectangles conditionally on hitting them, the probability that the graph of M does not hit the next one is lower bounded by $c/(\log n)^a$. This is the content of Proposition 2.1

2

in Section 2. In order to control the probabilities mentioned above, we also have to make sure that the two dimensional process visits enough new vertices in most intervals $[t_k, t_{k+1}]$. This is the content of Proposition 3.4 that we state and prove in Section 3.

In Section 4 we prove the following lemma, which shows that there is no dispersion result in the case $a = 1$, with general hypotheses like in Corollary 1.2.

Lemma 1.3. *There exists a positive constant c, such that for any n, there exists a martingale $(M_k)_{k \leq n}$ sastisfying almost surely*

$$\mathbb{E}\left[(M_{k+1} - M_k)^2 \mid \mathcal{F}_k\right] \geq 1 \quad and \quad |M_{k+1} - M_k| \leq \log n,$$

for all $k \leq n$, yet

$$\mathbb{P}(M_n = 0) \geq c.$$

Notation: For functions f, g we write $f(n) \lesssim g(n)$ if there exists a universal constant $C > 0$ such that $f(n) \leq Cg(n)$ for all n. We write $f(n) \gtrsim g(n)$ if $g(n) \lesssim f(n)$. Finally we write $f(n) \asymp g(n)$ if both $f(n) \lesssim g(n)$ and $f(n) \gtrsim g(n)$. We write $\mathcal{B}(x, r)$ to denote the ball in the $\mathcal{L}^1$- metric centered at x of radius r. Note also that in the rest of the paper we use c for a constant whose value may change from line to line.

2 Martingale defocusing

In this section we state and prove a dispersion result for martingales. Then in Section 3 we use it to prove our main result, Theorem 1.1 when $a = 1/2$.

We call the quadratic variation of a martingale M, the process $(V_t)_{t \geq 1}$ defined by

$$V_t = \sum_{\ell=1}^{t} \mathbb{E}\left[(M_\ell - M_{\ell-1})^2 \mid \mathcal{F}_{\ell-1}\right].$$

Proposition 2.1. *Let $\rho > 0$ be given. There exists a positive constant c and $n_0 \geq 1$ such that the following holds for any $a \in (0, 1)$. Suppose that M is a martingale with quadratic variation V and suppose that $(G_k)_k$ is an i.i.d. sequence of geometric random variables with mean 2 satisfying*

$$|M_{k+1} - M_k| \leq G_k \quad \forall k. \tag{2.1}$$

For each $1 \leq k < \log_2(n)$ we let $t_k = n - n/2^k$ and

$$A_k = \left\{V_{t_{k+1}} - V_{t_k} \geq \rho(t_{k+1} - t_k)/(\log n)^{2a}\right\}.$$

Suppose that for some $N \geq 1$ and $1 \leq k_1 < \ldots < k_N < \log_2(n)/2$, it holds

$$\mathbb{P}\left(\bigcap_{i=1}^{N} A_{k_i}\right) = 1. \tag{2.2}$$

Then we have for all $n \geq n_0$

$$\mathbb{P}(M_n = 0) \leq \exp\left(-cN/(\log n)^a\right).$$

Remark 2.2. We note that the choice of mean 2 for the geometric random variables in (2.1) is arbitrary. Any other value would be fine as well.

Proof of Corollary 1.2. If we divide the martingale M by $(\log n)^a$, then it satisfies the hypotheses of Proposition 2.1 with $N = \log_2(n)/2$, and hence the statement of the corollary follows. $\qquad\square$

Before proving Proposition 2.1 we state and prove a preliminary result that will be used in the proof later.

Lemma 2.3. *There exists $\rho > 0$ such that the following holds. Suppose that M is a martingale satisfying assumption (2.1) of Proposition 2.1. Let $m < \ell$ and $h > \log(\ell - m + 1)$ be given and let $\tau = \min\{t \geq m : |M_t - M_m| \geq h\}$. Suppose that $\mathbb{P}(V_\ell - V_m \geq h^2/\rho) = 1$. Then we have almost surely*

$$\mathbb{P}(\tau \leq \ell \mid \mathcal{F}_m) \geq \frac{1}{2}.$$

Proof. It is well known that the process $(M_t^2 - V_t)$ is a martingale. Since $\tau \wedge \ell \geq m$ is a stopping time, by the optional stopping theorem we get

$$\mathbb{E}\big[M_{\tau \wedge \ell}^2 - V_{\tau \wedge \ell} \mid \mathcal{F}_m\big] = M_m^2 - V_m. \tag{2.3}$$

Now we claim that

$$\mathbb{E}\big[M_{\tau \wedge \ell}^2 - M_m^2 \mid \mathcal{F}_m\big] \lesssim h^2. \tag{2.4}$$

Indeed we can write

$$\begin{aligned}
\mathbb{E}\big[M_{\tau \wedge \ell}^2 - M_m^2 \mid \mathcal{F}_m\big] &= \mathbb{E}\big[(M_{\tau \wedge \ell} - M_{(\tau-1)\wedge \ell} + M_{(\tau-1)\wedge \ell} - M_m)^2\big] \\
&\leq 2\mathbb{E}\big[(M_{\tau \wedge \ell} - M_{(\tau-1)\wedge \ell})^2\big] + 2h^2,
\end{aligned}$$

where the last inequality follows from the definition of τ. In order to bound the first term in the right hand side above, we use (2.1) and the fact that $\tau \geq m$. This way we get

$$|M_{\tau \wedge \ell} - M_{(\tau-1)\wedge \ell}| \leq \max_{m \leq t \leq \ell} |G_t|.$$

So we now obtain

$$\mathbb{E}\big[(M_{\tau \wedge \ell} - M_{(\tau-1)\wedge \ell})^2 \mid \mathcal{F}_m\big] \lesssim \log(\ell - m + 1)^2,$$

which proves our claim (2.4), using also the hypothesis $h > \log(\ell - m + 1)$. Since by assumption we have $\mathbb{P}(V_\ell - V_m \geq h^2/\rho) = 1$, we obtain that almost surely

$$\mathbb{E}[V_{\tau \wedge \ell} - V_m \mid \mathcal{F}_m] \geq \mathbb{E}[(V_\ell - V_m)\mathbf{1}(\tau \geq \ell) \mid \mathcal{F}_m] \geq \frac{h^2}{\rho}\mathbb{P}(\tau \geq \ell \mid \mathcal{F}_m).$$

This together with (2.3) and (2.4) and by taking ρ sufficiently small proves the lemma. $\qquad\square$

We are now ready to give the proof of Proposition 2.1.

Proof of Proposition 2.1. We will argue as in [10], by saying that in order for M_n to be at 0, the graph of M, i.e. the process $((t, M_t))_{t \leq n}$, has to cross the space-time rectangles H_k, for all $k = 1, \ldots, \log_2(n)$, which are defined by

$$H_k := [t_k, n] \times [-h_k, h_k] \quad \text{with} \quad h_k := \rho\sqrt{\frac{t_{k+1} - t_k}{(\log n)^{2a}}}. \tag{2.5}$$

4

We now define
$$\sigma_k = \inf\{t \ge t_k : |M_t| \ge h_k\}$$

For each $k \le \log_2(n)$ such that $\mathbb{P}(A_k) = 1$ we can apply Lemma 2.3 with $m = t_k$, $\ell = t_{k+1}$ and $h = 2h_k$ if n is sufficiently large so that $h > \log(\ell - m + 1)$. We thus deduce that for ρ sufficiently small and for all $n > n_0$ we have almost surely

$$\mathbb{P}(\sigma_k \le t_{k+1} \mid \mathcal{F}_{t_k}) \ge \frac{1}{2}. \tag{2.6}$$

Next we claim that a.s. conditionally on $\mathcal{F}_{\sigma_k}$ the martingale has probability of order $1/(\log n)^a$, to reach level $\pm h_k(\log n)^a/\rho^2$ before returning below level $\pm h_k/\sqrt{2} = \pm h_{k+1}$ (if at least one of these events occurs before time n). Indeed assume for instance that $M_{\sigma_k} \ge h_k$. Then the optional stopping theorem shows that on the event $E_k = \{M_{\sigma_k} \ge h_k\} \cap \{\sigma_k \le n\}$ we have

$$h_k \le M_{\sigma_k} = \mathbb{E}[M_{T_1 \wedge T_2 \wedge n} \mid \mathcal{F}_{\sigma_k \wedge n}], \tag{2.7}$$

where
$$T_1 := \inf\{t \ge \sigma_k \ : \ |M_t| \ge h_k(\log n)^a/\rho^2\},$$

and
$$T_2 := \inf\{t \ge \sigma_k \ : \ |M_t| \le h_k/\sqrt{2}\}.$$

We deduce from (2.7) that on E_k

$$h_k \le \mathbb{E}[M_{T_1}\mathbf{1}(T_1 < T_2 \wedge n) \mid \mathcal{F}_{\sigma_k}] + \frac{h_k(\log n)^a}{\rho^2}\mathbb{P}(n < T_1 \wedge T_2 \mid \mathcal{F}_{\sigma_k}) + \frac{h_k}{\sqrt{2}}.$$

Then by using again the bound (2.1) we get that on E_k

$$\begin{aligned}
\mathbb{E}[M_{T_1}\mathbf{1}(T_1 < T_2 \wedge n) \mid \mathcal{F}_{\sigma_k}] &\le \frac{h_k(\log n)^a}{\rho^2}\mathbb{P}(T_1 < T_2 \wedge n \mid \mathcal{F}_{\sigma_k}) + \mathbb{E}\left[\max_{t_k \le t \le n} G_t\right] \\
&\le \frac{h_k(\log n)^a}{\rho^2}\mathbb{P}(T_1 < T_2 \wedge n \mid \mathcal{F}_{\sigma_k}) + c_1\log(n - t_k + 1),
\end{aligned}$$

where c_1 is a positive constant. It follows that if n is large enough, then on E_k

$$\mathbb{P}(T_1 \wedge n < T_2 \mid \mathcal{F}_{\sigma_k}) \gtrsim \frac{1}{(\log n)^a}.$$

Similarly we get the same inequality with the event $\{M_{\sigma_k} \ge h_k\}$ replaced by $\{M_{\sigma_k} \le -h_k\}$, and hence we get that almost surely

$$\mathbb{P}(T_1 \wedge n < T_2 \mid \mathcal{F}_{\sigma_k})\,\mathbf{1}(\sigma_k \le n) \gtrsim \frac{1}{(\log n)^a}\mathbf{1}(\sigma_k \le n), \tag{2.8}$$

which proves our claim. We now notice that on the event $\{t_k \le T_1 \le n\}$ we have by Doob's maximal inequality

$$\mathbb{P}\left(\sup_{i \le n-t_k} |M_{i+T_1} - M_{T_1}| \ge h_k(\log n)^a/(2\rho^2) \ \middle|\ \mathcal{F}_{T_1}\right) \lesssim \frac{n - t_k}{(h_k(\log n)^a/\rho^2)^2} < c_1, \tag{2.9}$$

where c_1 is a constant that we can take smaller than 1 by choosing ρ small enough. Note that we used again (2.1) in order to bound the $\mathcal{L}^2$ norm of the increments of the martingale M.

Next we define a sequence of stopping times, which are the hitting times of the space-time rectangles (H_k) defined in (2.5). More precisely, we let $s_0 = 0$ and for $i \geq 1$ we let

$$s_i = \min\{t > s_{i-1} : (t, M_t) \in H_i\}.$$

Thus for each $k < \log_2(n)$ such that $\mathbb{P}(A_k) = 1$ by using (2.6), (2.8) and (2.9), we get that on the event $\{s_{k-1} \leq n\}$

$$\mathbb{P}\big(s_k > n \mid \mathcal{F}_{s_{k-1}}\big) \gtrsim \frac{1}{(\log n)^a}.$$

Using the assumption that the event $\cap_{i=1}^N A_{k_i}$ happens almost surely we obtain

$$\mathbb{P}(M_n = 0) \leq \mathbb{P}(s_{k_1}, \ldots, s_{k_N} \leq n) \leq \left(1 - \frac{c}{(\log n)^a}\right)^N \leq \exp\left(-cN/(\log n)^a\right),$$

for a positive constant c, and this concludes the proof. $\qquad \square$

3 Proof of transience

In this section we prove Theorem 1.1. We first give an equivalent way of viewing the random walk W. Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables taking values $(0, 0, \pm 1)$ equally likely. Let $\zeta_1, \zeta_2, \ldots$ be i.i.d. random variables taking values $(\pm 1, 0, 0), (0, \pm 1, 0)$ all with equal probability, and independent of the $(\xi_i)_i$. Assume that $(W_0, \ldots, W_t)$ have been defined, and set

$$r_W(t) = \#\{W_0, \ldots, W_t\}.$$

Then

$$W_{t+1} = \begin{cases} W_t + \xi_{r_W(t)} & \text{if } r_W(t) = r_W(t-1) + 1 \\ W_t + \zeta_{t-r_W(t)} & \text{otherwise} \end{cases}.$$

To prove Theorem 1.1 it will be easier to look at the process at the times when the two dimensional process moves. So we define a clock process $(\tau_k)_{k \geq 0}$ by $\tau_0 = 0$ and for $k \geq 0$,

$$\tau_{k+1} = \inf\{t > \tau_k : (X_t, Y_t) \neq (X_{\tau_k}, Y_{\tau_k})\} = \inf\{t > \tau_k : t - r_W(t) = k\}.$$

Note that $r_W(0) = 1$ and $\tau_k < \infty$ a.s. for all k. Observe that by definition the process $U_t := (X_{\tau_t}, Y_{\tau_t})$ is a $2d$-simple random walk, and that $r_W(\tau_t) = \tau_t - t + 1$. Note that

$$Z_t = \sum_{i=1}^{r_W(t)-1} \langle \xi_i, (0, 0, 1) \rangle.$$

We set $\mathcal{F}_t = \sigma(\xi_1, \ldots, \xi_{\tau_t - t})$, so that Z_{τ_t} is $\mathcal{F}_t$-measurable for all t.

We call $\mathbb{Q}$ the law of the process U. We denote by $\mathbb{P}_U()$ the law of the process W conditionally on the whole process U, or in other words on the whole sequence $(\zeta_i)_{i \geq 1}$. We write $\mathbb{P} = \mathbb{Q} \times \mathbb{P}_U$ for the law of the overall process W.

In the following claim we show that the process Z observed only at the times when the two-dimensional process moves, is a martingale.

Claim 3.1. *Let $M_t = Z_{\tau_t}$. Then $\mathbb{Q}$-a.s. we have that (M_t) is an $(\mathcal{F}_t)$-martingale under $\mathbb{P}_U$.*

Proof. We already noticed that M_t is adapted to $\mathcal{F}_t$ for all t. Now since the ξ_i's are i.i.d. and have mean 0 it follows from Wald's identity that

$$\mathbb{E}_U\left[Z_{\tau_{t+1}} \mid \mathcal{F}_t\right] = \mathbb{E}_U\left[Z_{\tau_t} + \sum_{i=\tau_t-t+1}^{\tau_{t+1}-t-1} \langle \xi_i, (0,0,1)\rangle \;\middle|\; \mathcal{F}_t\right] = Z_{\tau_t},$$

since $\tau_{t+1} - t - 1$ is the first time after $\tau_t - t + 1$ when we visit an already visited site, and is thus a stopping time. $\qquad\square$

Remark 3.2. We note that the jumps of the martingale are stochastically dominated by geometric random variables. More precisely, we can couple the process M (or W) with a sequence $(G_t)_{t\geq 0}$ of i.i.d. geometric random variables with parameter $1/2$, such that

$$|M_{t+1} - M_t| \leq G_t \quad \text{for all } t \geq 0. \tag{3.1}$$

Before proceeding, we give some more definitions. For $t \geq 0$, set

$$r_U(t) = \#\{U_0, \ldots, U_t\},$$

i.e. $r_U(t)$ is the cardinality of the range of the two-dimensional process up to time t. We also set for $t \geq 0$

$$V_t := \sum_{\ell=1}^{t} \mathbb{E}_U\left[(M_\ell - M_{\ell-1})^2 \mid \mathcal{F}_{\ell-1}\right].$$

Claim 3.3. *Suppose that U_ℓ is a fresh site, i.e. $U_\ell \notin \{U_0, U_1, \ldots, U_{\ell-1}\}$. Then*

$$\mathbb{E}_U\left[(M_{\ell+1} - M_\ell)^2 \mid \mathcal{F}_\ell\right] \geq 2.$$

Proof. Notice that when U_ℓ is a fresh site, then $M_{\ell+1} - M_\ell$ can be written as

$$M_{\ell+1} - M_\ell = \sum_{i=1}^{\tau} \lambda_i,$$

where $(\lambda_i)_i$ are i.i.d. random variables taking values ± 1 equally likely and

$$\tau = \inf\{k \geq 2 : (\lambda_{k-1}, \lambda_k) \in \{(-1, +1), (+1, -1)\}\}.$$

Then by the optional stopping theorem we deduce

$$\mathbb{E}_U\left[(M_{\ell+1} - M_\ell)^2 \mid \mathcal{F}_\ell\right] = \mathbb{E}\left[\left(\sum_{i=1}^{\tau} \lambda_i\right)^2\right] = \mathbb{E}[\tau] \geq 2,$$

since $\tau \geq 2$ by definition. $\qquad\square$

Before proving Theorem 1.1 we state a proposition that we prove later, which combined with the above claim guarantees that the quadratic variation V of the martingale M satisfies assumption (2.2) of Proposition 2.1. The following proposition only concerns the 2d-simple random walk.

Proposition 3.4. *For $k \geq 1$ we let $t_k = n - n/2^k$ and for $\rho > 0$ define*

$$\mathcal{K} = \left\{ 1 \leq k \leq (\log n)^{3/4} \ : \ r_U(t_{k+1}) - r_U(t_k) \geq \rho(t_{k+1} - t_k)/\log n \right\}.$$

Then there exist positive constants α, c and ρ_* such that for all $\rho < \rho_*$

$$\mathbb{P}\left(\#\mathcal{K} \leq \rho(\log n)^{3/4} \ \Big| \ U_n = 0 \right) \lesssim \exp(-c(\log n)^\alpha).$$

Proof of Theorem 1.1. Let $\mathcal{K}$ and ρ be as in Proposition 3.4. Note that $\mathcal{K}$ is completely deter-mined by the 2d-walk. Setting $A = \{\#\mathcal{K} \geq \rho(\log n)^{3/4}\}$ we then have

$$\mathbb{P}(U_n = M_n = 0) = \mathbb{E}[\mathbf{1}(U_n = 0)\mathbb{P}_U(M_n = 0)\,\mathbf{1}(A)] + \mathbb{E}[\mathbf{1}(U_n = 0)\mathbb{P}_U(M_n = 0)\,\mathbf{1}(A^c)]. \quad (3.2)$$

On the event A, using Claim 3.3 we get that there exist $k_1, \ldots, k_{\rho(\log n)^{3/4}} \in \mathcal{K}$ such that

$$\mathbb{P}_U\left(\bigcap_{i=1}^{\rho(\log n)^{3/4}} A_{k_i} \right) = 1,$$

where the events (A_i) are as defined in Proposition 2.1. We can now apply this proposition (with $a = 1/2$) to obtain

$$\mathbb{P}_U(M_n = 0)\,\mathbf{1}(\#\mathcal{K} \geq \rho(\log n)^{3/4}) \lesssim \exp(-c(\log n)^{1/4}).$$

Therefore from (3.2) we deduce

$$\mathbb{P}(U_n = M_n = 0) \lesssim \frac{1}{n}\exp(-c(\log n)^{1/4}) + \frac{1}{n}\exp(-c(\log n)^\alpha),$$

where α is as in Proposition 3.4. Since this last upper bound is summable in n, this proves that 0 is visited only finitely many times almost surely. Exactly the same argument would work for any other point of $\mathbb{Z}^3$, proving that W is transient. $\qquad \square$

Before proving Proposition 3.4 we state and prove a standard preliminary lemma and a corollary that will be used in the proof.

Lemma 3.5. *Let U be a simple random walk in $\mathbb{Z}^2$ starting from 0. Then there exists a positive constant c, such that for all $t \leq n\log n$ satisfying $\log(n/t) \lesssim (\log n)^{3/4}$ we have*

$$\mathbb{P}\left(\#\{U_0, \ldots, U_n\} \cap \mathcal{B}(0, \sqrt{t}) \geq \frac{t}{(\log n)^{1/16}} \right) \lesssim \exp(-c(\log n)^{1/16}).$$

Proof. To prove this we first decompose the path into excursions that the random walk makes across $\mathcal{B}(0, 2\sqrt{t}) \setminus \mathcal{B}(0, \sqrt{t})$ before time n. More precisely define $\sigma_0 = 0$, and for $i \geq 0$,

$$\sigma_i' = \inf\{k \geq \sigma_i \ : \ U_k \notin \mathcal{B}(0, 2\sqrt{t})\},$$

$$\sigma_{i+1} = \inf\{k \geq \sigma_i' \ : \ U_k \in \mathcal{B}(0, \sqrt{t})\}.$$

Let

$$N := \max\{i \ : \ \sigma_i \leq n\},$$

be the total number of excursions before time n, and for each $i \leq N$, let

$$R_i := \#\{U_{\sigma_i}, \ldots, U_{\sigma_i'}\},$$

be the number of points visited during the i-th excursion. Of course we have

$$\#\{U_0, \ldots, U_n\} \cap \mathcal{B}(0, \sqrt{t}) \leq \sum_{i=1}^{N} R_i. \tag{3.3}$$

Note that every time the random walk is on the boundary of the ball $\mathcal{B}(0, 2\sqrt{t})$, it has probability of order $1/(\log(n/t) + 4 \log \log n)$ to hit the boundary of the ball $\mathcal{B}(0, \sqrt{n} \log n)$ before hitting $\mathcal{B}(0, \sqrt{t})$ (see for instance [12]). If T is the first exit time from $\mathcal{B}(0, \sqrt{n}(\log n)^2)$, then

$$\mathbb{P}(T \leq n) \lesssim e^{-c(\log n)^4}, \tag{3.4}$$

where c is a positive constant. On the event $\{T \geq n\}$, it is easy to see that N is dominated by a geometric random variable with mean of order $\log(n/t)$. We thus get

$$\mathbb{P}\Big(N \geq (\log(n/t) + 4 \log \log n)(\log n)^{1/16}\Big)$$

$$\leq \mathbb{P}(T \leq n) + \mathbb{P}\Big(N \geq (\log(n/t) + 4 \log \log n)(\log n)^{1/16}, T \geq n\Big) \tag{3.5}$$

$$\lesssim \exp\left(-c(\log n)^4\right) + \exp\left(-c(\log n)^{1/16}\right) \lesssim \exp\left(-c(\log n)^{1/16}\right). \tag{3.6}$$

Since we have $\mathbb{E}_x[\sigma_i' - \sigma_i] \lesssim t$ for all $x \in \mathcal{B}(0, 2\sqrt{t})$, by using the Markov property we can deduce

$$\mathbb{P}\Big(\sigma_i' - \sigma_i \geq t(\log n)^{1/16}\Big) \leq \exp(-c(\log n)^{1/16}).$$

Moreover, it follows from [6, Lemma 4.3] and the fact that $\log n \asymp \log t$ that

$$\mathbb{P}\left(\#\{U_{\sigma_i}, \ldots, U_{\sigma_i + t(\log n)^{1/16}}\} \geq \frac{t}{(\log n)^{7/8}}\right) \leq \exp(-c(\log n)^{1/16}).$$

Combining the last two inequalities, we get that for any i,

$$\mathbb{P}\left(R_i \geq \frac{t}{(\log n)^{7/8}}\right) \leq 2 \exp(-c(\log n)^{1/16}),$$

where c is a positive constant. Using the assumption that $\log(n/t) \lesssim (\log n)^{3/4}$ together with (3.3) and (3.5) concludes the proof of the lemma. $\qquad\square$

Corollary 3.6. *Let U be a simple random walk in $\mathbb{Z}^2$, let $t \leq n$ satisfying $\log(n/(n-t)) \lesssim (\log n)^{3/4}$ and let $\varepsilon < 1/32$. Then there exists a positive constant c such that*

$$\mathbb{P}\left(\#\{U_0, \ldots, U_t\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n-t}) \geq \frac{n-t}{(\log n)^{\frac{1}{16} - 2\varepsilon}} \,\middle|\, U_n = 0\right) \lesssim \exp\left(-c(\log n)^{1/16}\right).$$

Proof. First we use the rough bound:

$$\#\{U_0, \ldots, U_t\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n-t}) \leq \#\{U_0, \ldots, U_{n/2}\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n-t})$$
$$+ \#\{U_{n/2}, \ldots, U_n\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n-t}).$$

9

We now note that if A is an event only depending on the first $n/2$ steps of the random walk, then we have

$$\mathbb{P}(A \mid U_n = 0) = \frac{\mathbb{P}(U_n = 0 \mid A)\,\mathbb{P}(A)}{\mathbb{P}(U_n = 0)} \lesssim \mathbb{P}(A)\,, \tag{3.7}$$

where the last inequality follows from the local central limit theorem. By reversibility we obtain

$$\mathbb{P}\left(\#\{U_{n/2}, \ldots, U_n\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n-t}) \geq \frac{n-t}{2(\log n)^{\frac{1}{16}-2\varepsilon}} \,\middle|\, U_n = 0 \right)$$

$$= \mathbb{P}\left(\#\{U_0, \ldots, U_{n/2}\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n-t}) \geq \frac{n-t}{2(\log n)^{\frac{1}{16}-2\varepsilon}} \,\middle|\, U_n = 0 \right). \tag{3.8}$$

The statement now readily follows by combining Lemma 3.5 with (3.7) and (3.8). $\qquad\square$

Proof of Proposition 3.4. Let us consider the events

$$A_k := \left\{ r_U(t_{k+1}) - r_U(t_k) \geq \rho\, \frac{t_{k+1} - t_k}{\log n} \right\},$$

with $\rho > 0$ some constant to be fixed later. Let also $\varepsilon < 1/48$,

$$B_k = \left\{ \#\{U_0, \ldots, U_{t_k}\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n-t_k}) \leq \frac{t_{k+1} - t_k}{(\log n)^{\frac{1}{16}-2\varepsilon}} \right\},$$

and

$$\widetilde{B}_k := B_k \cap \left\{ U_{t_k} \in \mathcal{B}(0, \sqrt{n-t_k}) \right\}.$$

Set for $k = 1, \ldots, (\log n)^{3/4}$,

$$\mathcal{G}_k = \sigma(U_0, \ldots, U_{t_k}),$$

and note that $\widetilde{B}_k \in \mathcal{G}_k$.

Claim 3.7. *For any $k \leq (\log n)^{3/4}$ we have almost surely*

$$\mathbb{P}(A_k^c \mid \mathcal{G}_k)\,\mathbf{1}(\widetilde{B}_k) \lesssim \frac{1}{(\log n)^\varepsilon}\,\mathbf{1}(\widetilde{B}_k).$$

Proof. To prove the claim we use two facts. On the one hand it follows from [5, Theorem 1.5] that if ρ is small enough, then a.s.

$$\mathbb{P}\left(\#\{U_{t_k+1}, \ldots, U_{t_{k+1}}\} \leq 2\rho\, \frac{t_{k+1} - t_k}{\log n} \,\middle|\, \mathcal{G}_k \right) \leq \exp\left(-c(\log n)^{1/6}\right).$$

Moreover, on the event $\{U_{t_k} \in \mathcal{B}(0, \sqrt{n-t_k})\}$, with probability at most $\exp(-c(\log n)^{2\varepsilon})$ the random walk exits the ball $\mathcal{B}(0, (\log n)^\varepsilon \sqrt{n-t_k})$ before time t_{k+1}. Therefore we obtain on the event $\{U_{t_k} \in \mathcal{B}(0, \sqrt{n-t_k})\}$ that

$$\mathbb{P}\left(\#\{U_{t_k+1}, \ldots, U_{t_{k+1}}\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n-t_k}) \leq 2\rho\, \frac{t_{k+1} - t_k}{\log n} \,\middle|\, \mathcal{G}_k \right) \lesssim \exp\left(-c(\log n)^{2\varepsilon}\right). \tag{3.9}$$

Suppose now on the other hand that a point is at distance at least $r = O(\sqrt{t})$ from U_{t_k}. Then it is well known (see for instance [12]) that the probability that the walk hits it during the time interval

10

$[t_k, t_k + t]$ is $O(\log(\sqrt{t}/r)/\log \sqrt{t})$. If we apply this with $t = t_{k+1} - t_k$, $r = \sqrt{n - t_k}/\log n$, and use that $\#\mathcal{B}(0, r) \asymp r^2$, we deduce that

$$
\mathbb{E}\big[\#\{U_0, \ldots, U_{t_k}\} \cap \{U_{t_k+1}, \ldots, U_{t_{k+1}}\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n - t_k}) \mid \mathcal{G}_k\big] \mathbf{1}(\widetilde{B}_k)
$$
$$
\lesssim \left(\frac{t_{k+1} - t_k}{(\log n)^2} + \frac{(t_{k+1} - t_k) \log\log n}{(\log n)^{17/16 - 2\varepsilon}} \right) \mathbf{1}(\widetilde{B}_k)
$$
$$
\lesssim \frac{t_{k+1} - t_k}{(\log n)^{1+\varepsilon}} \mathbf{1}(\widetilde{B}_k). \tag{3.10}
$$

We now have almost surely

$$
\mathbb{P}(A_k \mid \mathcal{G}_k) \mathbf{1}(\widetilde{B}_k) \geq \mathbb{P}\left(\#\{U_{t_k+1}, \ldots, U_{t_{k+1}}\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n - t_k}) > 2\rho \frac{t_{k+1} - t_k}{\log n} \,\Big|\, \mathcal{G}_k \right) \mathbf{1}(\widetilde{B}_k)
$$
$$
- \mathbb{P}\left(\#\{U_0, \ldots, U_{t_k}\} \cap \{U_{t_k+1}, \ldots, U_{t_{k+1}}\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n - t_k}) \geq \rho \frac{t_{k+1} - t_k}{\log n} \,\Big|\, \mathcal{G}_k \right) \mathbf{1}(\widetilde{B}_k)
$$
$$
\geq \left(1 - \exp\left(-c(\log n)^{1/6} \right) - \frac{c_1}{(\log n)^\varepsilon} \right) \mathbf{1}(\widetilde{B}_k),
$$

where the last inequality follows from (3.9), (3.10) and Markov's inequality. $\qquad\square$

Next, if we write $Q(\cdot) = \mathbb{P}(\cdot \mid U_n = 0)$ for the Doob transform of U, then $(U_k)_{k \leq n}$ is a Markov chain under Q. We let $\widetilde{A}_k = A_k \cap \{U_{t_{k+1}} \in \mathcal{B}(0, 2\sqrt{n - t_k})\}$. Then we have almost surely

$$
Q(A_k \mid \mathcal{G}_k) \mathbf{1}(\widetilde{B}_k) \geq Q\left(\widetilde{A}_k \,\Big|\, \mathcal{G}_k \right) \mathbf{1}(\widetilde{B}_k) \gtrsim \mathbb{P}\left(\widetilde{A}_k \,\Big|\, \mathcal{G}_k \right) \mathbf{1}(\widetilde{B}_k) \geq p \mathbf{1}(\widetilde{B}_k), \tag{3.11}
$$

where the penultimate inequality follows by the local central limit theorem as in (3.7) and the last inequality from Claim 3.7 and the fact that

$$
\mathbb{P}\big(U_{t_{k+1}} \in \mathcal{B}(0, 2\sqrt{n - t_k}) \mid U_{t_k} \in \mathcal{B}(0, \sqrt{n - t_k}) \big) \geq c > 0.
$$

Then we introduce the process $(M_k)_{k \leq (\log n)^{3/4}}$, defined by $M_1 = 0$ and for $k \geq 2$,

$$
M_k := \sum_{\ell=1}^{k-1} \left\{ \mathbf{1}(A_\ell \cap \widetilde{B}_\ell) - Q(A_\ell \mid \mathcal{G}_\ell) \mathbf{1}(\widetilde{B}_\ell) \right\}.
$$

Note that by construction it is a $(\mathcal{G}_k)$-martingale, under the measure Q. Since the increments of this martingale are bounded, it follows from Azuma-Hoeffding's inequality that for any $\kappa > 0$, there exists a positive constant c such that

$$
\mathbb{P}\left(|M_{(\log n)^{3/4}}| \geq \kappa (\log n)^{3/4} \,\Big|\, U_n = 0 \right) \lesssim \exp(-c(\log n)^{3/4}). \tag{3.12}
$$

As a consequence of Corollary 3.6 we get that

$$
1 - \mathbb{P}\left(\cap_{k \leq (\log n)^{3/4}} B_k \,\Big|\, U_n = 0 \right) \lesssim \exp(-c(\log n)^{1/16}). \tag{3.13}
$$

Claim 3.8. *There exists a positive constant c such that*

$$
\mathbb{P}\left(\sum_{k=1}^{(\log n)^{3/4}} \mathbf{1}(U_{t_k} \in \mathcal{B}(0, \sqrt{n - t_k})) \leq c(\log n)^{3/4} \,\Bigg|\, U_n = 0 \right) \leq \exp(-c(\log n)^{3/4}).
$$

Proof. By using reversibility and the local central limit theorem again, it suffices in fact to show the result without conditioning on $U_n = 0$, and replacing the times t_k by $n - t_k$. In other words, it suffices to prove that

$$\mathbb{P}\left(\sum_{k=1}^{(\log n)^{3/4}} \mathbf{1}(U_{2^k} \in \mathcal{B}(0, 2^{k/2})) \leq c(\log n)^{3/4} \right) \leq \exp(-c(\log n)^{3/4}), \qquad (3.14)$$

for some $c > 0$. This is standard, but for the sake of completeness we give a short proof now. We will prove in fact a stronger statement. Call

$$v_k := \inf\{t \geq 0 \ : \ U_t \notin \ \mathcal{B}(0, 2^{k/2})\}.$$

Obviously it is sufficient to prove (3.14) with the events $\{v_k > 2^k\}$ in place of $\{U_{2^k} \in \mathcal{B}(0, 2^{k/2})\}$. Set $\mathcal{H}_k = \sigma(U_0, \ldots, U_{v_k})$. Then it is well known that we can find a constant $\alpha > 0$, such that a.s. for any k,

$$\mathbb{P}(v_{k+1} > 2^{k+1} \mid \mathcal{H}_k) \geq \alpha.$$

Then by considering the martingale

$$M_k' := \sum_{\ell=1}^{k} \left(\mathbf{1}(v_\ell > 2^\ell) - \mathbb{P}(v_\ell > 2^\ell \mid \mathcal{H}_{\ell-1}) \right),$$

and using the Azuma-Hoeffding inequality the desired estimate follows. So the proof of the claim is complete. $\qquad \square$

By taking ρ and κ sufficiently small and using (3.11), (3.12), (3.13) and Claim 3.8 finishes the proof of the proposition. $\qquad \square$

4 Example

In this section we construct the martingale of Lemma 1.3. Before doing so, we recall a self-interacting random walk (X, Y, Z) in $\mathbb{Z}^3$ which was mentioned in [7] and is closely related to the random walk of Theorem 1.1; on the first visit to a vertex only (X, Y) performs a two-dimensional step, while on later visits to the same vertex only Z changes by ± 1 equally likely. Our proof in this case does not apply, or at least another argument is required. Indeed, by looking again at the process Z at the times when (X, Y) moves, we still obtain a martingale, but we do not have a good control on the jumps of this martingale. In particular, up to time n, they could be of size of order $\log n$, which might be a problem as Lemma 1.3 shows.

Proof of Lemma 1.3. Define $M_0 = 0$. Let $(S_k^i)_{k,i}$ be independent (over i) simple random walks on $\mathbb{Z}$ and let $(\widetilde{S}_k^i)_{k,i}$ be independent (over i) random walks with jumps that take values $\pm[\log n]$ equally likely and start from 0. Let k_* be the first integer such that $n/2^{k_*} \leq (\log n)^2$. We now let

$$M_k = S_k^1 \quad \text{for} \quad k \leq n/2.$$

We define t_1 by

$$n - t_1 = \frac{n}{2} + \inf\left\{ t \geq 0 : \left| M_{n/2} + \widetilde{S}_t^1 \right| \leq \log n \right\}.$$

If $t_1 \geq 0$, then we let

$$M_{k+n/2} = M_{n/2} + \widetilde{S}_k^1 \quad \text{for} \quad 0 \leq k \leq \frac{n}{2} - t_1.$$

If $t_1 < 0$, then we let

$$M_{k+n/2} = M_{n/2} + \widetilde{S}_k^1 \quad \text{for} \quad 0 \leq k \leq \frac{n}{2}.$$

Suppose that we have defined $t_\ell > 0$, we now define $t_{\ell+1}$ inductively. We let

$$M_{k+n-t_\ell} = M_{n-t_\ell} + S_k^{\ell+1} \quad \text{for} \quad 0 \leq k \leq \frac{t_\ell}{2}$$

and we also define $t_{\ell+1}$ by

$$n - t_{\ell+1} = n - \frac{t_\ell}{2} + \inf\left\{t \geq 0 : \left|M_{n-t_\ell/2} + \widetilde{S}_t^{\ell+1}\right| \leq \log n\right\}.$$

If $t_{\ell+1} \geq 0$, then we let

$$M_{k+n-t_\ell/2} = M_{n-t_\ell/2} + \widetilde{S}_k^{\ell+1} \quad \text{for} \quad 0 \leq k \leq \frac{t_\ell}{2} - t_{\ell+1}.$$

If $t_{\ell+1} < 0$, then we let

$$M_{k+n-t_\ell/2} = M_{n-t_\ell/2} + \widetilde{S}_k^{\ell+1} \quad \text{for} \quad 0 \leq k \leq \frac{t_\ell}{2}.$$

In this way we define the times t_ℓ for all $\ell \leq k^*$, unless there exists ℓ such that $t_\ell < 0$, in which case we set $t_m = 0$ for all $\ell + 1 \leq m \leq k^*$. If $t_{k^*} > 1$, then at time $n - t_{k^*} + 1$ if $d(M_{n-t_{k^*}}, 0) \neq 0$, then the martingale makes a jump of size $\pm d(M_{n-t_{k^*}}, 0)$ equally likely. If $d(M_{n-t_{k^*}}, 0) = 0$, then with probability $1/(\log n)^2$ it jumps to $\pm[\log n]$, while with probability $1 - 1/(\log n)^2$ it stays at 0. From time $n - t_{k^*} + 2$ until time n at every step with probability $1/(\log n)^2$ it jumps to $\pm[\log n]$, while with probability $1 - 1/(\log n)^2$ it stays at its current location.

By the definition of the martingale it follows that it satisfies the assumptions of the lemma. It only remains to check that there exists a positive constant c such that $\mathbb{P}(M_n = 0) > c$. We define the events

$$E = \{M_{n-t_{k^*}+1=0}\} \quad \text{and} \quad E' = \{M_\ell = 0, \text{ for all } \ell \in \{n - t_{k^*} + 2, \ldots, n\}\}.$$

We now have

$$\begin{aligned}
\mathbb{P}(M_n = 0) \geq &\mathbb{P}\big(t_1 > 0, \ldots, t_{k^*} > 0, M_{n-t_{k^*}} = 0, E, E'\big) \\
&+ \mathbb{P}\big(t_1 > 0, \ldots, t_{k^*} > 0, M_{n-t_{k^*}} \neq 0, E, E'\big).
\end{aligned} \tag{4.1}$$

By the definition of the times t_i, it follows that $t_{i+1} \leq t_i/2$, and hence we deduce that $t_i \leq n/2^i$, which implies that $t_{k^*} \leq n/2^{k^*} \leq (\log n)^2$. We now obtain

$$\begin{aligned}
\mathbb{P}\big(E, E' \mid t_1 > 0, \ldots, t_{k^*} > 0, M_{n-t_{k^*}} \neq 0\big) &\gtrsim \left(1 - \frac{1}{(\log n)^2}\right)^{(\log n)^2} \\
\mathbb{P}\big(E, E' \mid t_1 > 0, \ldots, t_{k^*} > 0, M_{n-t_{k^*}} = 0\big) &\geq \left(1 - \frac{1}{(\log n)^2}\right)^{(\log n)^2}.
\end{aligned} \tag{4.2}$$

Using the estimate for a simple random walk that if $h > 0$, then

$$\mathbb{P}_h(S_k > 0, \forall k \leq n) \lesssim \frac{h}{\sqrt{n}},$$

we get for a positive constant c_1 that

$$\mathbb{P}(t_{\ell+1} > 0 \mid t_\ell > 0) = 1 - \mathbb{P}\left(\inf\{t \geq 0 : |M_{n-t_\ell/2} + \widetilde{S}_t^{\ell+1}| \leq \log n\} \leq \frac{t_\ell}{2} \,\Big|\, t_\ell > 0\right)$$

$$\geq 1 - \frac{c_1}{\log n}.$$

Hence from (4.1) and (4.2) together with the above estimate and the fact that $k^* \asymp \log n$, we finally conclude

$$\mathbb{P}(M_n = 0) \gtrsim \left(1 - \frac{c_1}{\log n}\right)^{c_2 \log n} \cdot \left(1 - \frac{1}{(\log n)^2}\right)^{(\log n)^2} \geq c_3 > 0$$

and this finishes the proof of the lemma. $\qquad\square$

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