# Collisions of Random Walks

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**Abstract**

A recurrent graph $G$ has the infinite collision property if two independent random walks on $G$, started at the same point, collide infinitely often a.s. We give a simple criterion in terms of Green functions for a graph to have this property, and use it to prove that a critical Galton-Watson tree with finite variance conditioned to survive, the incipient infinite cluster in $\mathbb{Z}^d$ with $d \geq 19$ and the uniform spanning tree in $\mathbb{Z}^2$ all have the infinite collision property. For power-law combs and spherically symmetric trees, we determine precisely the phase boundary for the infinite collision property.

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## 1 Introduction

Let $G$ be an infinite connected recurrent graph, and let $X$ and $Y$ be independent (discrete time) simple random walks on $G$. For classical examples such as $\mathbb{Z}$ or $\mathbb{Z}^2$ it is easy to see that $X$ and $Y$ collide infinitely often – that is $Z = |\{t : X_t = Y_t\}| = \infty$, a.s. However, Krishnapur and Peres [19] gave an example (the graph Comb($\mathbb{Z}$) which is described below) of a recurrent graph for which the number of collisions $Z$ is a.s. finite. This had an element of surprise, as this graph is recurrent, whence the expected number of collisions is infinite, see the remarks following Theorem 1.1 of [19] . In this paper we study the finite collision property in more detail. We start by establishing a simple zero one law and a sufficient condition (in terms of Green functions) for infinite collisions. Using this we show that a critical Galton-Watson tree (conditioned to survive forever), the incipient infinite cluster in high dimensions, and the uniform spanning tree in two dimensions all have the infinite collision property.

We then examine subgraphs of Comb($\mathbb{Z}$) and investigate when they have the infinite collision property, and then conclude the paper by looking at a class of spherically symmetric trees.

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**Remark 1.1.** We note that all recurrent transitive graphs have the infinite collision property, since the number of collisions in this case follows a geometric distribution.

**Remark 1.2.** A transient graph of uniformly bounded vertex degree always has the finite collision property. There are examples of transient graphs with unbounded vertex degree with infinitely many collisions. For more on the transient case we refer the reader to [19].

We begin by defining what we mean by the finite/infinite collision property. Throughout this paper we will only consider connected graphs.

**Definition 1.3.** *Let $G$ be a graph, and $X$, $Y$ be independent (discrete time) simple random walks on $G$. We write $\mathbb{P}_{a,b}$ for the law of the process $((X_t, Y_t), t \in \mathbb{Z}_+)$ when $X_0 = a, Y_0 = b$. Let*

$$Z = \sum_{t=0}^{\infty} \mathbf{1}(X_t = Y_t)$$

*be the total number of collisions between $X$ and $Y$. If*

$$\mathbb{P}_{a,a}(Z < \infty) = 1 \quad \text{for all } a \in G \tag{1.1}$$

*then $G$ has the* **finite collision property**. *If*

$$\mathbb{P}_{a,a}(Z = \infty) = 1 \quad \text{for all } a \in G \tag{1.2}$$

*then $G$ has the* **infinite collision property.**

We will see below that these are the only two possibilities.

We recall the definition of Comb($\mathbb{Z}$):

**Definition 1.4.** *Comb($\mathbb{Z}$) is the graph with vertex set $\mathbb{Z} \times \mathbb{Z}$ and edge set*

$$\{[(x, n), (x, m)] : |m - n| = 1\} \cup \{[(x, 0), (y, 0)] : |x - y| = 1]\}$$

**Definition 1.5.** *Following [11], we define the wedge comb with profile $f$, denoted Comb($\mathbb{Z}, f$) to be the subgraph of Comb($\mathbb{Z}$) with vertex set*

$$V = \{(x, y) \in \mathbb{Z}^2 : 0 \le y \le f(x)\}$$

*and edge set the set of edges of Comb($\mathbb{Z}$) with vertices in $V$. We write Comb($\mathbb{Z}, \alpha$) for the wedge comb with profile $f(k) = k^\alpha$.*

In [11] it is proved that Comb($\mathbb{Z}, \alpha$) has the infinite collision property when $\alpha < 1/5$.

We have the following phase transition:

**Theorem 1.6.** *(a) If $\alpha \le 1$, then Comb($\mathbb{Z}, \alpha$) has the infinite collision property.*
*(b) If $\alpha > 1$, then Comb($\mathbb{Z}, \alpha$) has the finite collision property.*

We remark that the proofs of both (a) and (b) extend to the profiles of the form $f(x) = C|x|^\alpha$. Part (b) for $1 < \alpha < 2$ was also obtained independently by J. Beltran, D.Y. Chen, T. Mountford and D. Valesin (private communication).

**Remark 1.7.** This theorem shows that if the 'teeth' in the comb are large then the finite collision property will hold, while it fails if they are small. However, there is no simple monotonicity property for the finite collision property: Comb($\mathbb{Z}$) has the finite collision property but is a subgraph of $\mathbb{Z}^2$, which does not.

Further, we do not have any kind of 'bracketing' property for collisions: we have Comb$(\mathbb{Z}, 1) \subset$ Comb$(\mathbb{Z}, 2) \subset \mathbb{Z}^2 \subset$ Comb$(\mathbb{Z}^2)$; and of these Comb$(\mathbb{Z}, 1)$ and $\mathbb{Z}^2$ have the infinite collision property while the other two graphs have the finite collision property. (See [19] for the definition of Comb$(\mathbb{Z}^2)$, and the proof that it has the finite collision property.)

In Section 3 we will obtain a criterion, in terms of Green functions, or equivalently electrical resistance, for a graph to have the infinite collision property. Using this, we can show that several graphs arising in critical phenomena have the infinite collision property.

**Theorem 1.8.** *The following random graphs all have the infinite collision property:*
*(a) A critical Galton-Watson tree with finite variance conditioned to survive forever.*
*(b) The incipient infinite cluster for critical percolation in dimension $d \geq 19$.*
*(c) The Uniform Spanning Tree (UST) in $\mathbb{Z}^2$.*

For background on the critical Galton Watson tree conditioned to survive, see [16]. For background on the incipient infinite cluster and the UST, see [5] and [18], respectively. See Corollary 3.5 for a class of critical Galton-Watson trees with infinite variance.

Another graph for which we can prove the infinite collision property is the supercritical percolation cluster in $\mathbb{Z}^2$, see Theorem 3.6. This was proved independently by Chen and Chen [10].

Finally we examine some spherically symmetric trees.

**Definition 1.9.** *A tree is called* ***spherically symmetric*** *if every vertex at distance $n$ from the root has the same number of children. Let $(b_j)_j$ be a sequence of positive integers. We define the* ***spherically symmetric tree associated to the sequence*** *$(b_j)_j$ as follows: we attach a segment of length $b_0$ to the root $o$ . At the end of that segment we have a branch point with two branches, each of them having length $b_1$, and so on.*

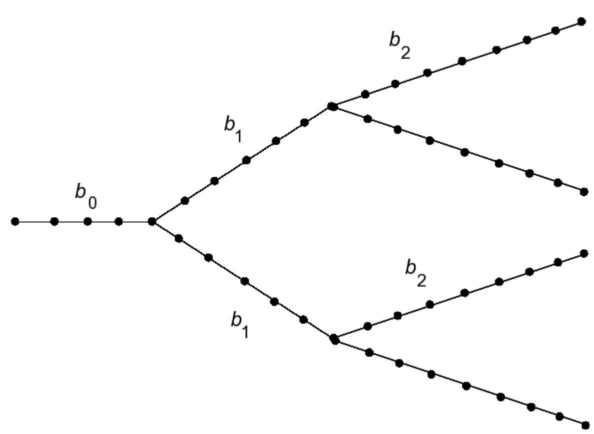


Figure 1. A spherically symmetric tree.

We will look at a class of spherically symmetric trees where the lengths are of the form $b_j = 2^{2^{\beta j}}$, where $\beta > 0$, and will show that these trees exhibit two phase transitions: the critical parameter for recurrence of the product chain on $T \times T$ is $\beta = 2$, while the critical parameter for the infinite collision property is $\beta = 1/2$. We establish this in the following Theorem.

**Theorem 1.10.** *(a) When $\beta \geq 2$, the product chain on $T \times T$ is recurrent, and hence the tree has the infinite collision property.*
*(b) When $\beta < 2$, the product chain on $T \times T$ is transient.*
*(c) When $\beta \geq \frac{1}{2}$, the tree has the infinite collision property.*
*(d) When $\beta < \frac{1}{2}$, the tree has the finite collision property.*

We use $c, c'c''$ to denote positive constants which may change on each appearance.

## 2 0-1 Law

In this section we are going to prove that the event of having infinitely many collisions in a recurrent graph is a trivial event, and hence has probability either 0 or 1. Thus in order to show the infinite collision property it suffices to show that infinitely many collisions occur with positive probability.

**Proposition 2.1.** *Let $G$ be a (connected) recurrent graph, $X$ and $Y$ be independent random walks on $G$, and $Z$ be the number of collisions. Then for each $(a, b) \in G \times G$,*

$$\mathbb{P}_{a,b}(Z = \infty) \in \{0, 1\}.$$

*Further, if there exist $a_0$, $b_0$ such that $\mathbb{P}_{a_0,b_0}(Z = \infty) > 0$ then $\mathbb{P}_{a,b}(Z = \infty) = 1$ for all $a$, $b$ such that $\mathbb{P}_{a,b}(X_m = a_0, Y_m = b_0) > 0$ for some $m \geq 0$. In particular, either $\mathbb{P}_{a,a}(Z = \infty) = 0$ for all $a$ or else $\mathbb{P}_{a,a}(Z = \infty) = 1$ for all $a$.*

*Proof.* Let $\mathcal{T}_n^X = \sigma(X_n, X_{n+1}, \dots)$, and define $\mathcal{T}_n^Y$ analogously. Then since $X$ is a recurrent Markov chain $\mathcal{T}^X = \cap_n \mathcal{T}_n^X$ is trivial by Orey's theorem (see [9]). By [21, Lemma 2] we have, since $X$ and $Y$ are independent, that

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(\mathcal{T}_n^X, \mathcal{T}_n^Y) = \sigma(\mathcal{T}^X, \mathcal{T}^Y),$$

which is trivial since $\mathcal{T}^X$ and $\mathcal{T}^Y$ are both trivial. Since the event $\{Y_n = X_n \text{ i.o.}\}$ is $\mathcal{T}$ measurable, it therefore has probability 0 or 1.

Now suppose $\mathbb{P}_{a_0,b_0}(Z = \infty) = 1$ and let $a, b, m$ be as above, i.e. $\mathbb{P}_{a,b}(X_m = a_0, Y_m = b_0) > 0$. Then

$$\mathbb{P}_{a,b}(Z = \infty) \geq \mathbb{P}_{a,b}(Z = \infty | X_m = a_0, Y_m = b_0)\mathbb{P}_{a,b}(X_m = a_0, Y_m = b_0) > 0.$$

By the 0-1 law therefore $\mathbb{P}_{a,b}(Z = \infty) = 1$. $\square$

**Remark 2.2.** The proof of Proposition 2.1 applies to any recurrent chain. Note that if $Z'$ denotes the total number of edges that are crossed at the same time by two independent random walks on a recurrent graph (started from the same state), then the event $\{Z' = \infty\}$ has probability zero or one (since the sequence of edges crossed by a recurrent random walk forms a recurrent chain.)

**Corollary 2.3.** *Let $A_n$ be finite subsets of $G$, let*

$$Z(A_n) := \sum_t \mathbf{1}(X_t = Y_t \in A_n)$$

*be the number of collisions in $A_n$, and $F_n = \{Z(A_n) > 0\}$.*
*(a) If $G = \cup_n A_n$ and $\mathbb{P}(F_n$ occurs i.o.$) = 0$ then $G$ has the finite collision property.*
*(b) If $A_n$ are disjoint and $\mathbb{P}(F_n) > c > 0$ for all $n$ then $G$ has the infinite collision property.*

*Proof.* (a) If $G \times G$ is recurrent then there are a.s. infinitely many collisions at each point $x \in G$, and so $\mathbb{P}(F_n$ occurs *i.o.*$) = 1$. We can therefore assume that $G \times G$ is transient. Hence there are only finitely many collisions in each set $A_n$, and as the total number of sets $A_n$ with a collision is finite, the total number of collisions is finite.
(b) We have $\mathbb{P}(F_n$ occurs i.o.$) > c$. However, $Z \geq \sum_n \mathbf{1}_{F_n}$, and so $\mathbb{P}(Z = \infty) > c$. So by the 0-1 law, Proposition 2.1, we get $\mathbb{P}(Z = \infty) = 1$. □

# 3 Green function criterion for $\infty$ collisions

## 3.1 Background material

Firstly we are going to recall a few facts about heat kernels and effective resistances. We will follow rather closely the exposition in [3] and [4]. Let $d(x)$ denote the degree of a vertex $x$ in a graph $G$. For two functions $f$, $g \in \mathbb{R}^{V(G)}$ we define the quadratic form

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{\substack{x,y \in V(G) \\ x \sim y}} (f(x) - f(y))(g(x) - g(y)).$$

We define the transition density

$$q_t(x, y) = \frac{\mathbb{P}_x(Y_t = y)}{d(y)}, \quad t \in \mathbb{Z}_+.$$

Here we have divided by the degree of the vertex to make the transition density a symmetric function.

Let $A$ and $B$ be two subsets of $V(G)$. The effective resistance between $A$ and $B$ is defined as follows:

$$R_{\text{eff}}(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : \mathcal{E}(f, f) < \infty, f|_A = 1, f|_B = 0\}. \tag{3.1}$$

The term effective resistance comes from electrical network theory, since we can think of our graph as an electrical network having unit resistances wherever there is an edge between two vertices. If we glue all points of $A$ to a point $a$ and all points of $B$ to $b$ and apply a voltage $V$ which then induces a current $I$ from $a$ to $b$, then the ratio $\frac{V}{I}$ is constant and is equal to the effective resistance.

From the definition (3.1) of effective resistance we see that there is a unique function $f$ achieving the infimum appearing on the right hand side of (3.1). This function must be harmonic everywhere outside the sets $A$ and $B$.

For any graph $G$ the effective resistance satisfies $R_{\text{eff}}(x,y) \leq d(x,y)$ and if $G$ is a tree, then

$$R_{\text{eff}}(x,y) = d(x,y),$$

where $d(x,y)$ stands for the graph-theoretic distance between $x$ and $y$.

Let $B(x_0,r) = \{y : d(x_0,y) \leq r\}$ and $Y_t^B$ ($B := B(x_0,r)$) be the discrete time simple random walk on $G$ killed when it exits $B(x_0,r)$ and let $q_t^B$ be its transition density. The Green kernel is defined by $g_B(x,y) = \sum_{t=0}^{\infty} q_t^B(x,y)$.

It is easy to see that $g_B(x,\cdot)$ is a harmonic function on $B \setminus \{x\}$ and that it satisfies the reproducing property, i.e. that $\mathcal{E}(g_B(x,\cdot),f) = f(x)$, for any function $f$ satisfying $f|_{B^c} = 0$.

The function defined by $h(y) := \frac{g_B(x,y)}{g_B(x,x)}$ is harmonic on $B \setminus \{x\}$ and takes value 1 at $x$ and 0 on $B^c$, hence $R_{\text{eff}}(x,B^c)^{-1} = \mathcal{E}(h,h)$. Using now the reproducing property mentioned above we get that

$$R_{\text{eff}}(x,B^c) = g_B(x,x),$$

a very useful equality that will be widely used in this paper.

If $B$ is a finite subset of $G$ then by spectral theory we can write

$$q_t^B(x,y) = \sum_i \lambda_i^t \varphi_i(x)\varphi_i(y), \tag{3.2}$$

where $\varphi_i$ are the eigenfunctions and $\lambda_i$ the eigenvalues of the associated transition operator. Since $|\lambda_i| \leq 1$ for all $i$ it follows that

$$q_{2t+1}^B(x,x) \leq q_{2t}^B(x,x) \quad \text{ for all } x \in B,\ t \geq 0. \tag{3.3}$$

Letting $B \uparrow G$ this inequality extends to $q_t$. From (3.3) we obtain

$$2g_B(x,x) \geq 2\sum_{t=0}^{\infty} q_{2t}^B(x,x) \geq \sum_{t=0}^{\infty}(q_{2t}^B(x,x) + q_{2t+1}^B(x,x)) = g_B(x,x). \tag{3.4}$$

## 3.2 The criterion

**Theorem 3.1.** *Let $G$ be a recurrent graph with a distinguished vertex $o$. Let $(B_r)_r$ be an increasing sequence of sets such that $B_r \neq G, \forall r$, and $\cup_r B_r = G$. Suppose that there exists $C < \infty$ such that for all $r$*

$$g_{B_r}(x,x) \leq C g_{B_r}(o,o), \quad \text{for all } x \in B_r,$$

*Then $G$ has the infinite collision property. Moreover, the number of edges crossed at the same time by two independent random walks is infinite a.s.*

*Proof.* Let $(B_r)_r$ be the sequence of sets satisfying the assumptions of the theorem. Set $B := B_r$ and let $X^B$ and $Y^B$ be the two random walks killed after exiting the set $B$, and let $q_t^B$ be their transition densities. Let $\widetilde{Z}_B$ count the number of edges that are crossed at the same time by these two random walks, i.e.

$$\widetilde{Z}_B = \sum_{t=0}^{\infty} \mathbf{1}(X_t^B = Y_t^B, X_{t+1} = Y_{t+1}).$$

To prove the theorem we are going to apply the second moment method to the random variable $\widetilde{Z}_B$, so we begin by computing its first and second moments. For the first moment we have

$$\begin{aligned}
\mathbb{E}_{o,o}[\widetilde{Z}_B] &= \sum_t \sum_{x \in B} \sum_{y \sim x} \mathbb{P}_{o,o}(X_t^B = Y_t^B = x, X_{t+1} = Y_{t+1} = y) \\
&= \sum_t \sum_{x \in B} \sum_{y \sim x} q_t^B(o,x)^2 d(x)^2 q_1(x,y)^2 d(y)^2 \\
&= \sum_t \sum_{x \in B} q_t^B(o,x)^2 d(x) = \sum_{t=0}^{\infty} q_{2t}^B(o,o).
\end{aligned}$$

We therefore have

$$g_B(o,o) \geq \mathbb{E}_{o,o}[\widetilde{Z}_B] \geq \frac{1}{2} g_B(o,o). \tag{3.5}$$

Observe that since $B_r \neq G$ and $G$ was assumed to be a recurrent graph, we have that $g_{B_r}(o,o) < \infty$.

And for the second moment we have

$$\begin{aligned}
\mathbb{E}_{o,o}[\widetilde{Z}_B^2] &= \mathbb{E}_{o,o}[\widetilde{Z}_B] + 2 \sum_t \sum_{s \geq t+1} \sum_{x \in B} \sum_{y \sim x} \sum_{z \in B} \sum_{w \sim z} q_t^B(o,x)^2 d(x)^2 q_1(x,y)^2 d(y)^2 \\
&\quad \times q_{s-t-1}^B(y,z)^2 d(z)^2 q_1(z,w)^2 d(w)^2 \\
&= \mathbb{E}_{o,o}[\widetilde{Z}_B] + 2 \sum_t \sum_{s \geq t+1} \sum_{x \in B} \sum_{y \sim x} \sum_{z \in B} q_t^B(o,x)^2 q_{s-t-1}^B(y,z)^2 d(z) \\
&\leq \mathbb{E}_{o,o}[\widetilde{Z}_B] + 2 \sum_t \sum_{x \in B} \sum_{y \sim x} q_t^B(o,x)^2 g_B(y,y) \\
&\leq g_B(o,o) + 2 g_B(o,o) \max_{y \in B} g_B(y,y).
\end{aligned} \tag{3.6}$$

Applying the second moment method to the variable $\widetilde{Z}_{B_r}$, and using (3.5), (3.6) and the hypotheses of the theorem we obtain

$$\mathbb{P}_{o,o}(\widetilde{Z}_{B_r} > \frac{1}{2} \mathbb{E}_{o,o}[\widetilde{Z}_{B_r}]) \geq \frac{1}{4} \frac{(\mathbb{E}_{o,o}[\widetilde{Z}_{B_r}])^2}{\mathbb{E}_{o,o}[\widetilde{Z}_{B_r}^2]} \geq \frac{g_B(o,o)}{16(1 + 2C g_B(o,o))}.$$

Since $g_{B_r}(o,o) \geq d(0)^{-1}$, it follows that $\mathbb{P}_{o,o}(\widetilde{Z}_{B_r} > \frac{1}{4} g_{B_r}(o,o)) \geq c > 0$, for all $r > 0$. As $r \to \infty$ we have $\widetilde{Z}_{B_r} \nearrow \widetilde{Z}$, where $\widetilde{Z}$ is the total number of common edges traversed by $X$ and $Y$. Letting $r \to \infty$ , we get $\mathbb{P}_{o,o}(\widetilde{Z} = \infty) > c$. Since $Z \geq \widetilde{Z}$, we have $\mathbb{P}_{o,o}(Z = \infty) > c$, and so by the 0-1 law, Proposition 2.1, we get $\mathbb{P}_{o,o}(Z = \infty) = 1$. For the last statement use Remark 2.2. □

The proof of (3.5) also gives

**Lemma 3.2.** *Suppose that $d(x) \le D$ for all $x \in G$. Let $Z_B$ be the total number of collisions of the killed walks $X^B$ and $Y^B$. Then*

$$\frac{1}{2} g_B(o,o) \le \mathbb{E}_{o,o} Z_B \le D g_B(o,o).$$

### 3.3 Applications of the Green kernel criterion

We now give a number of applications of this criterion, and in particular will prove Theorem 1.6(a) and Theorem 1.8.

*Proof of Theorem 1.6 (part **a**).* Let $B := B_r$ denote the set of vertices that are on the right of the origin and at horizontal distance at most $r$ from it – see Figure 2 below. Then $g_{B_r}(0,0) = R_{\text{eff}}(0, B_r^c) = d(0, B_r^c) = r+1$ and since $\alpha \le 1$ we have that $g_{B_r}(x,x) = R_{\text{eff}}(x, B_r^c) = d(x, B_r^c) \le r+1 = g_{B_r}(0,0)$, for any $x \in B_r$. □

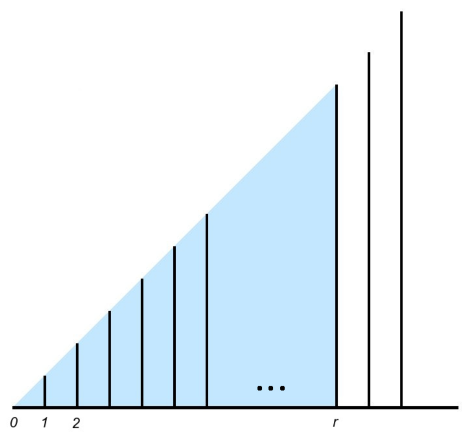


Figure 2. The set $B_r$ in the wedge comb.

*Proof of Theorem 1.8(a).* In this proof we have two types of randomness. We define the Galton-Watson tree on a probability space $(\Omega, P)$, and denote the tree $G(\omega)$, and its root $o$.

Let us quickly recall the structure of the critical Galton-Watson tree with finite variance conditioned to survive forever. For more details see for instance [16]. Let $(p_k)$ be the offspring distribution of the critical Galton Watson tree. Now start with the root $o$. Give it a random number of offspring which follows the size-biased distribution, i.e. $P(X = k) = kp_k$. The random variable $X$ has finite expectation, since the original distribution $p_k$ has finite variance. Choose one of its offspring at random and give it a random number of offspring with the size-biased distribution independently of before, and to all the others attach critical Galton Watson trees with the same offspring distribution $(p_k)$.

From this construction it follows that there is a unique infinite line of descent, which we call the **backbone** and off the nodes on it there are critical finite trees emanating.

Let $B_r$ be the set of vertices on the backbone that are at distance at most $r$ from the root, taken together with all their descendants that are off the backbone – see Figure 3.

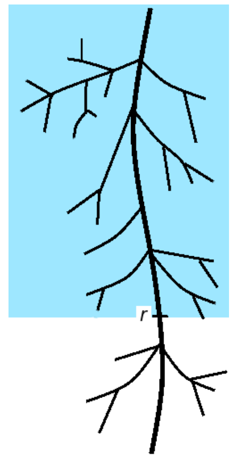

Figure 3. A Galton-Watson tree with the set $B_r$.

Fix $\varepsilon > 0$. Let $N_r^\varepsilon$ be the number of trees of depth greater than $\frac{r}{\varepsilon}$ that are contained in the set $B_r$, excluding the backbone itself. If $(Z_n)$ is a critical branching process with finite variance, then Kolmogorov's theorem states that

$$P(Z_n > 0) \sim \frac{2}{n\sigma^2} \text{ as } n \to \infty. \tag{3.7}$$

Let $Y_i$, for $i = 0, \cdots, r$, be the number of offspring of the $i$-th vertex on the backbone excluding the offspring on the backbone. Then $E(Y_i) = \sum_{k=1}^{\infty} k^2 p_k - 1 = \sigma^2$. We label the offspring of the $i-$th vertex on the backbone by $j = 1, \cdots, Y_i$ if $Y_i \geq 1$. Also, we let $T_{i,j}$, for $j = 1, \cdots, Y_i$, be the descendant tree of the $j$-th child off the backbone. Using (3.7) we have

$$P(N_r^\varepsilon \geq 1) \leq E(N_r^\varepsilon) = E\left(\sum_{i=0}^{r}\sum_{j=1}^{Y_i} \mathbf{1}\left(T_{i,j} \text{ has depth } > \frac{r}{\varepsilon}\right)\right) \leq \sum_{i=1}^{r} E[Y_i]\left(\frac{c\varepsilon}{\sigma^2 r}\right) \leq c\varepsilon,$$

where $c$ is a positive constant. So $P(N_r^\varepsilon = 0) \geq 1 - c\varepsilon$ and by Fatou's lemma we have, setting $A_\varepsilon = \{\omega : N_r^\varepsilon(\omega) = 0 \text{ i.o.}\}$, that

$$P(A_\varepsilon) = P(N_r^\varepsilon = 0 \text{ i.o.}) \geq \limsup_r P(N_r^\varepsilon = 0) \geq 1 - c\varepsilon.$$

Now $g_{B_r}(o,o) = r + 1$, and if $N_r^\varepsilon = 0$ then $g_{B_r}(x,x) \leq r + r/\varepsilon$ for all $x \in B_r$. If $\omega \in A_\varepsilon$ then applying the Green kernel criterion to the sets $B_r$ with $r$ being such that $N_r^\varepsilon(\omega) = 0$, we get the infinite collision property for the graph $G(\omega)$. Hence we deduce that

$$P(G(\omega) \text{ has the infinite collision property }) \geq P(A_\varepsilon) \geq 1 - c\varepsilon,$$

and thus sending $\varepsilon \to 0$, we get that $G$ has the infinite collision property $P$-a.s. $\square$

We have the following easy corollary of Theorem 3.1

**Corollary 3.3.** *Let $(G(\omega))$ be a family of random graphs (defined on a space $(\Omega, P)$) with a distinguished vertex $o$, and let $B_r = B(o, r)$. For $\lambda \geq 1$ let*

$$J(\lambda) = \{r \in \mathbb{Z}_+ : R_{\text{eff}}(o, B_r^c) \geq r/\lambda\}.$$

*Suppose that there exists a function $\psi(\lambda)$ with $\lim_{\lambda\to\infty} \psi(\lambda) = 0$ and $r_0 \geq 1$ such that*

$$P(r \in J(\lambda)) \geq 1 - \psi(\lambda) \quad \textit{for all } r \geq r_0. \tag{3.8}$$

*Then $G$ has the infinite collision property $P$-a.s.*

*Proof.* For each $x \in B_r$ we have $g_{B_r}(x, x) \leq d(x, o) + r \leq 2r$, while for each $r \in J(\lambda)$

$$g_{B_r}(o, o) = R_{\text{eff}}(o, B_r^c) \geq r/\lambda.$$

The condition (3.8) implies that

$$P(r \in J(\lambda) \text{ for infinitely many } r) \geq 1 - \psi(\lambda).$$

If this event holds then Theorem 3.1 implies that $G$ has the infinite collision property. Letting $\lambda \to \infty$ concludes the proof. □

*Proof of Theorem 1.8 (b) and (c).* For both of these graphs the condition (3.8) has been verified. For the incipient infinite cluster in dimension $d \geq 19$ see the proof of (2.1) at the end of section 2 of [18]. For the UST see Proposition 3.6 of [5]. □

**Remark 3.4.** We could also have used Corollary 3.3 to prove Theorem 1.8(a), since [14, Proposition 1.1] proves that a critical Galton-Watson tree with finite variance conditioned to survive forever satisfies the condition (3.8). However, we preferred to give a simple direct proof.

We can also use Corollary 3.3 to handle a class of critical Galton-Watson trees with infinite variance.

**Corollary 3.5.** *Let $(Z_n)$ be a critical Galton-Watson process with infinite variance such that*

$$\mathbb{E}[s^{Z_1}] = s + (1 - s)^\alpha L(1 - s),$$

*where $\alpha \in (1, 2]$, and $L(t)$ is slowly varying as $t \to 0$. Let $T^*$ be the tree associated with the process $(Z_n)$ conditioned to survive forever. Then $T^*$ has the infinite collision property.*

*Proof.* The condition (3.8) for this tree is proved in [12, Lemma 3.1]. □

We also have that many 'fractal' graphs satisfy the infinite collision property. Examples of graphs of this kind are given in Examples 3 and 4 in Section 5 of [3]: these include the graphical Sierpinski gasket – see Figure 1 in [18]. All these graphs have bounded vertex degree, and there exist $\beta \geq 2$ and $\alpha \in [\beta - 1, \beta)$ such that for $x, y \in G$, $r \geq 1$

$$|B(x, r)| \asymp r^\alpha, \qquad R_{\text{eff}}(x, y) \asymp d(x, y)^{\beta - \alpha}.$$

Lemma 2.2 of [3] then proves that

$$R_{\text{eff}}(x, B(x,r)^c) \geq c r^{\beta-\alpha}.$$

We therefore have

$$\max_{y\in B(x,r)} \frac{g_{B(x,r)}(y,y)}{g_{B(x,r)}(x,x)} = \max_{y\in B(x,r)} \frac{R_{\text{eff}}(y, B(x,r)^c)}{R_{\text{eff}}(x, B(x,r)^c)} \leq C$$

for all $x \in G$, $r \geq 1$. Hence the hypotheses of Theorem 3.1 hold, and the graph has the infinite collision property.

Now we are going to give a short proof of the following theorem, which was proved independently in [10].

**Theorem 3.6.** *Let $X$ and $Y$ be two independent discrete time simple random walks on the infinite supercritical percolation cluster in $\mathbb{Z}^2$ started from the same point. Then $X$ and $Y$ will collide infinitely many times a.s.*

*Proof.* Let $P_p^*$ denote the bond percolation measure $P_p$ in $\mathbb{Z}^2$ conditioned on the origin being in the infinite open cluster $\mathcal{C}_\infty$, and let $B_n$ be the $n$ by $n$ box centered at the origin in $\mathbb{Z}^2$. Denote by $\mathcal{C}_n$ the component of the origin in $\mathcal{C}_\infty \cap B_n$. It is well known, see e.g. [15], that

$$P_p^*(\mathcal{C}_n \text{ is the largest open cluster in } B_n) \to 1 \text{ as } n \to \infty.$$

Thus if $R_n$ denotes the maximal effective resistance between two nodes in the largest open cluster in $B_n$, and $\widetilde{R}_n$ is the maximum over $x \in \mathcal{C}_n$ of the effective resistance in $\mathcal{C}_n$ between $x$ and $\partial B_n$, then Rayleigh's monotonicity principle yields that $P_p^*(\widetilde{R}_n \leq R_n) \to 1$ as $n \to \infty$. In [7, Corollary 3.1] it is proved that for a large enough constant $A$, we have $P_p(R_n > A\log n) \to 0$ as $n \to \infty$, whence also $P_p^*(R_n > A\log n) \to 0$. We deduce that $P_p^*(\widetilde{R_n} > A\log n) \to 0$. But Rayleigh's monotonicity principle implies that in $\mathcal{C}_n$ we have $R_{\text{eff}}(0, \partial B_n) \geq a\log n$ for a suitable $a > 0$. Applying the Green kernel criterion, Theorem 3.1, establishes the infinite collision property in $\mathcal{C}_\infty$. □

# 4 Wedge comb with $\alpha > 1$

In Section 3 we proved that wedge combs with profile $f(x) = x^\alpha$ where $\alpha \leq 1$ have the infinite collision property. In this section we will prove Theorem 1.6(b), that is that if $\alpha > 1$ then the wedge comb has the finite collision property. We do not have any simple general criterion for the finite collision property, and our proofs will rely on making sufficiently accurate estimates of the transition density $q_t(x,y)$.

For $x \in G$ we write $x_1$ for the first coordinate of $X$.

Throughout this section we set

$$\alpha' = \alpha \wedge 2, \quad \beta' = \frac{1+\alpha'}{2+\alpha'}.$$

Note that $1 \leq \alpha' \leq 2$ and $2/3 \leq \beta' \leq 3/4$.

The main work in this section will be in proving the following.

**Lemma 4.1.** *Let $x = (k, h) \in G$. The transition density $q$ satisfies:*

$$q_t(0, x) \leq \begin{cases} \frac{c}{t^{\beta'}} & \text{if } t \geq k^{2+\alpha'}, \\ \frac{c}{(k^{2+\alpha'})^{\beta'}} & \text{if } t < k^{2+\alpha'}. \end{cases} \tag{4.1}$$

**Remark 4.2.** We note that the constants used in the proofs in this section do not depend on $t$, but could depend on $\alpha$.

Before we prove this Lemma, we will show how it leads easily to Theorem 1.6(b).

We define the set $Q_{k,h}$, where $h \leq k^{\alpha}$, as follows:

$$Q_{k,h} = \{(k, y) : 0 \leq y \leq h\},$$

and we set $Z_{k,h} = Z(Q_{k,h})$ to be the number of collisions of the two random walks in $Q_{k,h}$. We also define $\tilde{Z}_{k,h} = Z_{k,2h/3} - Z_{k,h/3}$, i.e. the number of collisions that happen in the set $\{(k, y) : \frac{h}{3} \leq y \leq \frac{2h}{3}\}$.

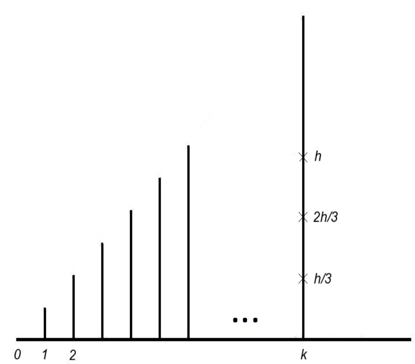


Fig 4

**Lemma 4.3.** *(a) $\mathbb{E}[Z_{k,h}] \leq chk^{-\alpha'}$.*
*(b) $\mathbb{E}[Z_{k,h} | \tilde{Z}_{k,h} > 0] \geq ch$.*

*Proof.* (a) By Lemma 4.1 we have

$$\mathbb{E}[Z_{k,h}] = \sum_t \sum_{x \in Q_{k,h}} q_t(0, x)^2 = \sum_{t < k^{2+\alpha'}} h \frac{c}{k^{2(1+\alpha')}} + \sum_{t \geq k^{2+\alpha'}} \frac{ch}{t^{2\beta'}} \leq \frac{ch}{k^{\alpha'}}.$$

(b) Since we are conditioning on the event $\{\tilde{Z}_{k,h} > 0\}$, there is a collision at position $x = (k, y)$ for some $y$ with $\frac{h}{3} \leq y \leq \frac{2h}{3}$. Conditioned on this event, the total number of collisions that happen in the set $Q_{k,h}$ will be greater than the number of collisions that take place before the first time that one of the random walks exits this interval. So, setting $B := Q_{k,h}$, and using (3.5) we have

$$\mathbb{E}[Z_{k,h} | \tilde{Z}_{k,h} > 0] \geq \frac{1}{2} g_{Q_{k,h}}(x, x) = \frac{1}{2} R_{\text{eff}}(x, Q^c_{k,h}) \geq ch.$$

$\square$

*Proof of Theorem 1.6(b).* By Lemma 4.3

$$chk^{-\alpha'} \geq \mathbb{E}[Z_{k,h}] \geq \mathbb{P}(\tilde{Z}_{k,h} > 0)\mathbb{E}[Z_{k,h}|\tilde{Z}_{k,h} > 0] \geq ch\mathbb{P}(\tilde{Z}_{k,h} > 0),$$

so that $\mathbb{P}(\tilde{Z}_{k,h} > 0) \leq ck^{-\alpha'}$. Now summing over all $k$ and over all $h$ ranging over powers of 2 and satisfying $h \leq k^\alpha$, we get that

$$\sum_k \sum_{h \text{ powers of } 2} \mathbb{P}(\tilde{Z}_{k,h} > 0) \leq \sum_k \log_2(k^\alpha)ck^{-\alpha'} < \infty, \text{ since } \alpha' > 1.$$

Hence by Corollary 2.3 the total number of collisions is finite almost surely. □

Before we prove Lemma 4.1 we give some ***heuristics*** for the bound $\mathbb{E}[Z_{k,h}] \leq chk^{-(\alpha\wedge 2)}$:

The expected time that the random walk takes to reach $k$ on the horizontal axis started from 0 is of the order $k^{2+\alpha}$. The reason for that is that the expected number of visits by the first coordinate to $i \in \mathbb{Z}_+$ before hitting $k$ for the first time is $k - i$. At every such visit the walk makes a vertical excursion, which takes time of order $i^\alpha$ in expectation. Hence the total time has expectation which is of order $k^{2+\alpha}$. Another way to see this is that the hitting time is half the commute time (due to reversibility) which is given by the resistance times the volume. For all $\alpha > 1$, we have that $k^{2+\alpha}$ is the right order for the expected time. The actual time though differs in the two regimes $1 < \alpha < 2$ and $\alpha > 2$.

The first coordinate makes $k^2$ steps to go from $\frac{k}{2}$ to $k$. When $1 < \alpha < 2$, at each step of the horizontal coordinate we perform an independent experiment. We succeed in each experiment, if we spend time greater than $k^{2\alpha}$ on the tooth in this step of the first coordinate. The probability of success is then lower bounded by $\frac{c_1}{k^\alpha}$ and in the $k^2$ experiments with high probability there will be a success and the expected number of successes is $k^{2-\alpha}$, thus the total time taken to reach $k$ will be of order $k^{2+\alpha}$.

When $\alpha > 2$ the experiments described above will give us no success with high probability, and so this method no longer gives us the right order for the hitting time. In this regime instead we declare a success if we spend time greater than $k^4$ on the tooth. The expected number of successes is then 1 and thus the total time to reach $k$ is of order $k^4$.

Thus the relevant times that will contribute to the expectation of $Z_{k,h}$ will be of order $k^{2+\alpha'}$. The probability that the two random walks will have the same horizontal coordinate will be $\left(\frac{1}{k}\right)^2$ and the probability that they will be at the right height will be $\left(\frac{h}{k^{\alpha'}}\right)^2$ and at the same height will be $\frac{1}{h}$. We get the uniform distribution, because by that time the random walks will have mixed.

Putting all things together in the formula for the expectation we obtain the aforementioned expression.

The remainder of this section is devoted to the proof of Lemma 4.1. Our main tool to bound $q_t(0, x)$ will be by comparison with Greens functions.

**Lemma 4.4.** *Let $B \subset G$. Then*

$$q_t(x, x) \leq \frac{2g_B(x, x)}{t\mathbb{P}_x(\tau_B \geq t)}. \tag{4.2}$$

*Proof.* The spectral decomposition (3.2) shows that $q_{2j}(x,x)$ is decreasing as a function of $j$, and also that $q_{2j+1}(x,x) \le q_{2j}(x,x)$ for $j \ge 0$. Using this it is easy to verify that

$$q_t(x,x) \le \frac{2}{t} \sum_{j=0}^{t} q_j(x,x). \tag{4.3}$$

We now define $g_t(x,x)$ to be the Green kernel until time $t$, i.e. $g_t(x,x) = \sum_{j=0}^{t} q_j(x,x)$. By the strong Markov property and the fact that $g_t(y,x) \le g_t(x,x)$ for all $y$ we get

$$g_t(x,x) \le g_B(x,x) + \mathbb{P}(\tau_B < t) g_t(x,x),$$

where $\tau_B$ is the first exit time from the set $B$; rearranging gives (4.2). □

To use this lemma we wish to choose the set $B$ so that the Green kernel up to time $t$ and the Green kernel of the Markov chain killed after exiting the set $B$ are comparable. To obtain the necessary bounds on the exit times from the region $B$ we now make precise some of the heuristics given above.

Note that for notational convenience we will often write $\mathbb{P}_k$ instead of $\mathbb{P}_{(k,0)}$.

**Lemma 4.5.** *(a) Let $k \ge 0$, $k_1 \ge 1$ and $T = \tau_{H(k-k_1,k+k_1)}$ be the first exit of $X$ from $H(k-k_1,k+k_1)$, where $H(a,b) := \{(x,y) \in G : a \le x \le b\}$. Then*

$$\mathbb{P}_k(T \le t) \le c \exp(-c(k_1^{2+\alpha'}/t)^{1/3}). \tag{4.4}$$

*(b) Let $k \ge 1$ and $T = \tau_{H(0,k)}$. Then*

$$\mathbb{P}_0(T \le t) \le c \exp(-c(k^{2+\alpha'}/t)^{1/3}).$$

*Proof.* Note that (b) follows from (a) by just looking at the random walk from the first hit on $2k/3$.

Suppose we have (4.4) when $k_1 \le k$. Then if $k_1 > k$ we have $H(k-k_1,k+k_1) = H(0,k+k_1)$, and $\frac{k+k_1}{2} \ge k$. Then since $X$ has to hit $\frac{k+k_1}{2}$ before it leaves $H(0,k+k_1)$, we have

$$\mathbb{P}_k(T \le t) \le \mathbb{P}_{\frac{k+k_1}{2}}(T \le t) \le c \exp(-c(k_1^{2+\alpha'}/t)^{1/3}).$$

Thus it is sufficient to consider the case when $k_1 \le k$.

We now prove (a) in the case when $k_1 \le k$. Let $L$ be the number of horizontal steps that the random walk makes until it leaves $H = H(k-k_1/2, k+k_1/2)$. Choose constants $\lambda > 0$ and $\theta \le \frac{1}{4}$. We have

$$\mathbb{P}_k(T \le t) \le \mathbb{P}_k(L < k_1^2/\lambda) + \mathbb{P}_k(T \le t, L \ge k_1^2/\lambda). \tag{4.5}$$

The first probability appearing on the right hand side of (4.5) is bounded above by the probability that a simple random on $\mathbb{Z}_+$ travels distance $k_1/2$ in less than $k_1^2/\lambda$ steps, which is smaller than $c' \exp(-c''\lambda)$.

To bound the second probability we are going to perform $N = k_1^2/\lambda$ independent experiments. In each experiment we succeed if we hit level $\theta k_1^{\alpha'}$ on the tooth, and then spend time at least $\theta^2 k_1^{2\alpha'}$ in the tooth before the next horizontal step. (The conditions $\theta \le \frac{1}{4}$ and $k_1/2 \le k/2$ ensure that there is enough room in each tooth.) Since a simple random walk on $\mathbb{Z} \cap [0, n]$ started at $m \le n$ has probability at least $c_1$ of taking more than $m^2$ steps to hit zero, the probability of success for each experiment is at least $p = c_1/(\theta k_1^{\alpha'})$. Thus on the event $\{L \ge N\}$ we have that $T$ stochastically dominates $\theta^2 k_1^{2\alpha'} \operatorname{Bin}(k_1^2/\lambda, p)$.

Hence

$$\mathbb{P}_k(T \le t, L \ge k_1^2/\lambda) \le \mathbb{P}\left(\operatorname{Bin}(k_1^2/\lambda, p) \le \frac{t}{\theta^2 k_1^{2\alpha'}}\right) = \mathbb{P}\left(\operatorname{Bin}(N, p) \le s\right), \tag{4.6}$$

where $s = t/(\theta^2 k_1^{2\alpha'})$.

By a straightforward application of Chernoff's bound we have:

**Lemma 4.6.** *Let $\mu < 1$. Then there exists a positive constant $\mu'$ such that*

$$\mathbb{P}(\operatorname{Bin}(n, p) \le \mu n p) \le e^{-\mu' n p}.$$

Define$\gamma$ by $t = \gamma k_1^{2+\alpha'}$. If $\gamma^{2/3} > (8c_1)^{-1}$ then by adjusting the constants $c$ the bound (4.4) holds. We can therefore assume that $\gamma^{2/3} \le (8c_1)^{-1}$. Let $\lambda = \gamma^{-1/3}$, and $\theta = (2/c_1)\gamma\lambda = (2/c_1)\gamma^{2/3}$; note that we have $\theta \le \frac{1}{4}$. Then if

$$\mu = \frac{s}{Np} = \frac{\lambda t}{c_1 \theta k_1^{2+\alpha'}} = \frac{\gamma\lambda}{c_1\theta} = \frac{1}{2},$$

Lemma 4.6 gives

$$\mathbb{P}_k(T < t, L \ge k_1^2/\lambda) \le e^{-cNp} \le \exp(-c k_1^{2-\alpha'}(c_1^2/2)\gamma^{-1/3}) \le e^{-c'\gamma^{-1/3}}. \tag{4.7}$$

Thus both terms in (4.5) are bounded by terms of the form $c\exp(-c'\gamma^{-1/3})$. ◻

**Lemma 4.7.** *$q_t(u, u) \le \frac{c}{t^{\beta'}}$ for any $u = (k, 0)$ on the horizontal axis and $t \ge 1$.*

*Proof.* Let $k_1 = bt^{1/(\alpha'+2)}$, where $b \ge 1$ is a constant which will be chosen later. We use Lemma 4.4 with

$$B = H(k - k_1, k + k_1) = \{(x, y) \in G : k - k_1 \le x \le k + k_1\}.$$

Then $R_{\text{eff}}(x, B^c) \le c k_1$. By Lemma 4.5

$$\mathbb{P}_k(\tau_B < t) \le c_1 \exp(-c_2(k_1^{2+\alpha'}/t)^{1/3}) = c_1 \exp(-c_2 b^{(2+\alpha')/3}). \tag{4.8}$$

Taking $b$ large enough, the right side of (4.8) can be made less than $1/2$. Hence by Lemma 4.4

$$q_t(u, u) \le ct^{-1} R_{\text{eff}}(u, B^c) \le ct^{-1} k_1 \le c' t^{-\beta'}.$$

◻

**Corollary 4.8.** *Let $u = (k, 0)$ on the horizontal axis and $t \geq 1$. Then*

$$q_t(0, u) \leq \frac{c}{t^{\beta'}}.$$

*Proof.* If $t$ is even, then by the Cauchy-Schwarz inequality and Lemma 4.7 we get

$$q_t(0, u) \leq \sqrt{q_t(0,0)}\sqrt{q_t(u,u)} \leq \frac{c}{t^{\beta'}}.$$

If $t$ is odd, then by the same argument we have

$$q_t(0, u) = q_{t-1}((1,0), u) \leq \frac{c}{t^{\beta'}}.$$

□

**Lemma 4.9.** $q_t(0, (k, h)) \leq ct^{-\beta'} e^{-\frac{h^2}{c't}}$, *for all points $(k, h)$ and all times $t > 0$.*

*Proof.* Let $T_A$ be the first hitting time of the set $A$ for a simple random walk $(S)$ on $\mathbb{Z}$. Using the ballot theorem (see, e.g. [17]) we get

$$\mathbb{P}_h(T_0 = s) = \frac{h}{s}\mathbb{P}_h(S_s = 0) \leq c\frac{h}{s}\frac{1}{\sqrt{s}}e^{-\frac{h^2}{c's}}. \tag{4.9}$$

Let $T_a^b$ be the first hitting time of a for a simple random walk restricted to the interval $[a, b]$. Then for all $s$ we have

$$\begin{aligned}\mathbb{P}_h(T_0^m = s) &= \mathbb{P}_{h-m}(T_{-m}^0 = s) = \mathbb{P}_{h-m}(T_{\{-m,m\}} = s) \\ &\leq \mathbb{P}_{h-m}(T_{-m} = s) + \mathbb{P}_{h-m}(T_m = s) \leq \mathbb{P}_h(T_0 = s) + \mathbb{P}_{2m-h}(T_0 = s). \end{aligned} \tag{4.10}$$

Thus by (4.9) we deduce

$$\mathbb{P}_h(T_0^m = s) \leq c\frac{h}{s}\frac{1}{\sqrt{s}}e^{-\frac{h^2}{c's}} + c\frac{2m-h}{s}\frac{1}{\sqrt{s}}e^{-\frac{(2m-h)^2}{c's}}. \tag{4.11}$$

By reversibility we have $q_t(0, (k, h)) = q_t((k, h), 0)$ and so by Corollary 4.8

$$q_t((k,h),0) \leq \sum_{s=1}^{t-1} \mathbb{P}_h(T_0^{k^\alpha} = s) q_{t-s}((k,0),0) \leq \sum_{s=1}^{t-1} \mathbb{P}_h(T_0^{k^\alpha} = s) c(t-s)^{-\beta'}.$$

Therefore we get

$$\begin{aligned} q_t((k,h),0) &\leq hc\left(\sum_{s=1}^{\frac{t}{2}} \frac{1}{s^{\frac{3}{2}}}\frac{1}{t^{\beta'}}e^{-\frac{h^2}{c's}} + \sum_{\frac{t}{2}\leq s\leq t-1} \frac{1}{t^{\frac{3}{2}}}\frac{1}{(t-s)^{\beta'}}e^{-\frac{h^2}{c's}}\right) \\ &\quad + (2k^\alpha - h)c\left(\sum_{s=1}^{\frac{t}{2}} \frac{1}{s^{\frac{3}{2}}}\frac{1}{t^{\beta'}}e^{-\frac{(2k^\alpha-h)^2}{c's}} + \sum_{\frac{t}{2}\leq s\leq t-1} \frac{1}{t^{\frac{3}{2}}}\frac{1}{(t-s)^{\beta'}}e^{-\frac{(2k^\alpha-h)^2}{c's}}\right) \\ &\leq hc''e^{-\frac{h^2}{c't}}\frac{1}{t^{\beta'}\sqrt{t}} + (2k^\alpha - h)c''e^{-\frac{(2k^\alpha-h)^2}{c't}}\frac{1}{t^{\beta'}\sqrt{t}}. \end{aligned}$$

Since $xe^{-\frac{x^2}{c't}} \leq c_1\sqrt{t}e^{-\frac{x^2}{2c't}}$ for all $x$ and $2k^\alpha - h \geq h$, we obtain

$$q_t((k,h),0) \leq c_1 t^{-\beta'} e^{-\frac{h^2}{c_1 t}}.$$

□

**Lemma 4.10.** *Let $x = (k, 0)$. Then if $t < k^{2+\alpha}$, we have*

$$q_t(0, x) \leq \frac{c}{k^{1+\alpha}} = c\left(k^{2+\alpha}\right)^{-\beta}.$$

*Hence*

$$\sup_{t \geq 0} q_t(0, x) \leq c\left(k^{2+\alpha}\right)^{-\beta}.$$

*Proof.* Let $m$ be an integer within distance 1 of $k/2$, and let $T = T_m$. Now

$$\mathbb{P}_0(X_t = x) = \mathbb{P}_0(X_t = x, T_m \leq t/2) + \mathbb{P}_0(X_t = x, T_m \geq t/2). \tag{4.12}$$

By time reversibility, we have that

$$\mathbb{P}_0(X_t = x, T_m > t/2) = c\mathbb{P}_x(X_t = 0, \text{ last visit to } m \text{ before } t/2) \leq c\mathbb{P}_x(X_t = 0, T_m < t/2),$$

so to bound the second term in (4.12) it suffices to bound $\mathbb{P}_x(X_t = 0, T_m \leq t/2)$.

By the strong Markov property we have

$$\mathbb{P}_0(X_t = x, T_m \leq t/2) \leq \mathbb{P}_0(T_m \leq t/2) \max_{0 \leq s \leq t/2} \mathbb{P}_m(X_{t-s} = x).$$

We bound the first term above using Lemma 4.5, while the second term is bounded by $ct^{-\beta}$. Thus writing $t = k^{2+\alpha}/\eta$, we have

$$\begin{aligned}\mathbb{P}_0(X_t = x, T_m \leq t/2) &\leq ct^{-\beta} \exp(-c(k^{2+\alpha}/t)^{1/3}) \\ &\leq ck^{-1-\alpha}\eta^{\beta} e^{-c\eta^{1/3}} \\ &\leq ck^{-1-\alpha} \sup_{\eta > 0}(\eta^{\beta} e^{-c\eta^{1/3}}) \leq c' k^{-1-\alpha}.\end{aligned}$$

The term $\mathbb{P}_x(X_t = 0, T_m \leq t/2)$ is bounded in exactly the same way. □

*Proof of Lemma 4.1.* Let $x = (k, h)$. Then the bound follows from Lemma 4.9 if $t \geq k^{2+\alpha'}$. If $t \leq k^{2+\alpha'}$ then by considering the first hit on $k$

$$\mathbb{P}_x(X_t = 0) \leq \max_{0 \leq s \leq t} \mathbb{P}_k(X_s = 0) \leq ck^{-1-\alpha'}$$

by Lemma 4.10. □

# 5 Spherically symmetric trees

In this section we are going to show the double phase transition taking place in the spherically symmetric trees of lengths $b_n = 2^{2^{\beta n}}$. We remark that for part (c) the Green kernel criterion, Theorem 3.1 does not apply.

*Proof of Theorem 1.10(a).* Let $(X_n, Y_n)_n$ be a discrete time walk on the product space $T \times T$. To show recurrence of the pair $(X_n, Y_n)$, we are going to use the Nash-Williams criterion of recurrence, which can be found for instance in [20, Chapter 21, Proposition 21.6].

Let $\Pi_n = \{x \in T : d(0,x) = n\}$, and

$$\Pi_n^* = (\Pi_n \times \cup_{i \le n} \Pi_i) \cup (\cup_{i \le n} \Pi_i \times \Pi_n) \subset T \times T.$$

Let $E_n$ be the set of edges with at least one vertex in $\Pi_n^*$. Then the sets $(E_{2n})_n$ constitute a sequence of disjoint edge-cutsets that separate $(o,o)$ from $\infty$. To show recurrence of $(X_n, Y_n)_n$, by the Nash-Williams criterion we only need to show that

$$\sum_n |E_{2n}|^{-1} = \infty. \tag{5.1}$$

We have $|E_n| \le c|\Pi_n^*| \le c'|\Pi_n| \times (\sum_{i=1}^n |\Pi_i|)$. However $|\Pi_n| \asymp (\log n)^{\frac{1}{\beta}}$, and so $\sum_{i=1}^n |\Pi_i| \asymp \sum_{i=1}^n (\log i)^{\frac{1}{\beta}} \le n (\log n)^{\frac{1}{\beta}}$. Hence

$$|\Pi_n^*| \le Cn(\log n)^{\frac{2}{\beta}},$$

and therefore as $\beta \ge 2$ (5.1) diverges. $\square$

The remaining parts of the proof will require estimates of the transition probabilities of the random walk $X$ on $T$. Since it will sometimes be convenient to use these rather than the transition density $q_t(x,y)$ we write

$$p_t(x,y) = \mathbb{P}_x(X_t = y).$$

Let

$$a_n = \sum_{i=0}^{n-1} b_i, \quad n \ge 1. \tag{5.2}$$

Note that the $n$-th branch point from $o$ is at distance $a_n$ from $o$. For $x \in T$ let $n(x)$ be the number of branches at the same level as $x$; $n(x)$ is also the number of vertices $y \in T$ such that $d(o,y) = d(o,x)$. We write

$$J_n = \{x \in T : n(x) = 2^n\};$$

these are the points between the $(n-1)$ -th and $n$-th branch points.

We now divide the segment of length $b_n$ into subintervals. The first one has length equal to $2^0 a_n$, the second one $2a_n$ and the $\ell$-th one has length $2^{\ell-1} a_n$. In total we get order $2^{\beta(n-1)}$ such intervals, say $\alpha 2^{\beta(n-1)}$. Let $I_{n,\ell}^i$ denote the $\ell$-th such interval on the $i$-th branch, for $i = 1, \cdots, 2^n$ and let $J_{n,\ell}$ denote the collection of all these subintervals, i.e. $2^n$ in total.

**Remark 5.1.** Our main tool will be by comparison with a birth and death chain $\tilde{X}$ on $\mathbb{Z}_+$ that jumps to either $x+1$ or $x-1$ with the following probabilities. If $x$ is at distance $a_n$ from the origin for some $n \ge 1$, then $p_{x,x+1} = \frac{2}{3} = 1 - p_{x,x-1}$, otherwise for all other $x$, $p_{x,x+1} = \frac{1}{2} = 1 - p_{x,x-1}$. Note that $(n(x))$ is a stationary measure for this birth and death chain. We write $p_t^{\mathrm{BD}}(0,x')$ for the transition probabilities and $q_t^{\mathrm{BD}}(0,x')$ for its transition density with respect to its stationary measure $(n(x))$. Note that $X'_t = d(o, X_t)$ has the law of this birth and death chain. Therefore if for $x \in T$ we write $|x| = d(o,x)$ then by symmetry

$$p_t(o,x) = \frac{1}{2^{n(x)}} p_t^{\mathrm{BD}}(0,|x|). \tag{5.3}$$

We write $\tau'$, $T'$ etc. for hitting and exit times for the birth and death chain.

**Lemma 5.2.** ***(a)*** *For the birth and death chain we have for all $t \geq 0$*

$$p_t^{BD}(0,0) \leq ct^{-1/2}(\log t)^{-1/\beta} \tag{5.4}$$

*and also*

$$p_t^{BD}(y,x) \leq \frac{c(\log x)^{1/\beta}}{\sqrt{t}(\log t)^{1/\beta}}, \quad \forall x, y \in \mathbb{Z}_+. \tag{5.5}$$

***(b)*** *For $x \in (2^\ell a_n, 2^{\ell+1} a_n)$ and times $t > (2^\ell a_n)^2$ we have*

$$p_t^{BD}(0,x) \geq \frac{c}{\sqrt{t}}. \tag{5.6}$$

***(c)*** *Finally for $x \in (2^\ell a_n, 2^{\ell+1} a_n)$ and times $t \leq (2^\ell a_n)^2$ we have*

$$p_t^{BD}(0,x) \leq \frac{c}{2^\ell a_n}. \tag{5.7}$$

*Proof.* **(a)** Let $B = \{y \in \mathbb{Z}_+ : y \leq |x| + \sqrt{t}\}$. Applying Lemma 4.4 to the birth and death chain $X'$

$$q_t^{\mathrm{BD}}(|x|,|x|) \leq \frac{c'' R_{\mathrm{eff}}^{\mathrm{BD}}(|x|, B^c)}{t \mathbb{P}_{|x|}(\tau'_B \geq t)}.$$

It is easy to verify that $R_{\mathrm{eff}}^{\mathrm{BD}}(|x|, B^c) \leq 2^{-n(x)-m} 2\sqrt{t}$, where $m$ is the number of branch points between $|x|$ and $|x| + \sqrt{t}$; note that at each branch point the effective resistance is halved. Since there are approximately $\beta^{-1} \log_2 \log_2 r$ branch points between 0 and $r$, we have $m = \frac{1}{\beta}\left(\log_2 \log_2(|x| + \sqrt{t}) - \log_2 \log_2 |x|\right)$, if $x \neq o$ and $\frac{1}{\beta} \log_2 \log_2(\sqrt{t})$ if $x = o$. We now need to bound $\mathbb{P}_{|x|}(\tau'_B < t)$. Since for each $n$, $\sum_{k=1}^n b_k \asymp 2^{2^{\beta n}}$, there must exist a branch of length at least $\frac{1}{2}\sqrt{t}$ between $|x|$ and $|x| + \sqrt{t}$; call this branch $A$. Let $y$ be the midpoint of $A$. Then $\mathbb{P}_{|x|}(\tau'_B < t)$ is smaller than the probability that a simple random walk started at $y$ remains in $A$ for time at least $t$. But from the exponential hitting time bounds for the simple random walk on $\mathbb{Z}$ we get that $\mathbb{P}_{|x|}(\tau'_B \geq t) > 1/2$ and hence since $2^{n(x)} \asymp (\log_2 |x|)^{1/\beta}$,

$$q_t^{\mathrm{BD}}(|x|,|x|) \leq c'' 2^{-n(x)} t^{-1/2} \left(\frac{\log_2(|x| + \sqrt{t})}{\log_2 |x|}\right)^{-1/\beta} \leq c'' t^{-1/2} \left(\log_2 t\right)^{-1/\beta}. \tag{5.8}$$

Similarly

$$q_t^{\mathrm{BD}}(0,0) \leq c'' t^{-1/2} \left(\log(c_2 \sqrt{t})\right)^{-1/\beta}.$$

We have that $p_t^{\mathrm{BD}}(y,x) = q_t^{\mathrm{BD}}(y,x) 2^{n(x)}$. So by Cauchy Schwartz we get that $p_t^{\mathrm{BD}}(y,x) \leq 2^{n(x)} \sqrt{q_t^{\mathrm{BD}}(y,y)} \sqrt{q_t^{\mathrm{BD}}(x,x)}$ and thus using (5.8) we obtain that $p_t^{\mathrm{BD}}(y,x) \leq \frac{c(\log x)^{\frac{1}{\beta}}}{\sqrt{t}(\log t)^{1/\beta}}$, (**b**) We write

$$\begin{aligned} p_t^{\mathrm{BD}}(0,x) = \mathbb{P}_0(T_x < t, X_t = x) &\geq \mathbb{P}_0(T_x < t) \min_{2s \leq t} p_{2s}(x,x) \\ &\geq \mathbb{P}_0(T_x < t)(p_t^{\mathrm{BD}}(x,x) + p_{t-1}^{\mathrm{BD}}(x,x)), \end{aligned}$$

since $p_{2s}(x,x)$ is a decreasing function of $s$. Let $Q_t = \{y \in \mathbb{Z}_+ : y \leq x + c\sqrt{t}\}$. Then by Cauchy-Schwartz we have

$$\begin{aligned} p_{2t}^{\mathrm{BD}}(x,x) &= \sum_y p_t^{\mathrm{BD}}(x,y) p_t^{\mathrm{BD}}(y,x) = \sum_y p_t^{\mathrm{BD}}(x,y)^2 \frac{n(x)}{n(y)} \\ &\geq \sum_{y \in Q_t} p_t^{\mathrm{BD}}(x,y)^2 \frac{n(x)}{n(y)} \geq \frac{n(x)}{|Q_t|} \left( \sum_{y \in Q_t} \frac{p_t^{\mathrm{BD}}(x,y)}{\sqrt{n(y)}} \right)^2. \end{aligned}$$

For $y \in Q_t$ we have $n(y) \leq \left(\log(x + c\sqrt{t})\right)^{\frac{1}{\beta}}$. Also $|Q_t| = x + c\sqrt{t} \leq c_1\sqrt{t}$, so

$$p_{2t}^{\mathrm{BD}}(x,x) \geq \frac{(\log x)^{\frac{1}{\beta}}}{c_1\sqrt{t}\left(\log(c_1\sqrt{t})\right)^{\frac{1}{\beta}}} \mathbb{P}_x(X_t \in Q_t)^2$$

and $\mathbb{P}_x(X_t \in Q_t) > c' > 0$ by the same argument we used in part(a) of the proof, i.e. by bounding it a by simple random walk on the last segment with no branch points. Also $\mathbb{P}_0(T_x < t) \geq \frac{1}{2}$, since $t > 2x^2$ and we can bound the birth and death chain from below by a simple random walk on $\mathbb{Z}_+$.

(**c**) Let $z$ and $y$ be two points on $\mathbb{Z}_+$ which are at even distance apart and such that $y \leq z$. Suppose that we start two birth and death chains $X$ from $z$ and $Y$ from $y$ and we couple them in such a way that $X_t \geq Y_t$ for all $t$ before the first time that they meet and after that time $X_t = Y_t$. From this coupling it follows immediately that

$$p_t^{\mathrm{BD}}(z,0) \leq p_t^{\mathrm{BD}}(y,0).$$

If there is no branch point between $z$ and $y$, then we get the same inequality, i.e. $p_t^{\mathrm{BD}}(0,z) \leq p_t^{\mathrm{BD}}(0,y)$. If there is one branch point between them, then we get $p_t^{\mathrm{BD}}(0,z) \leq 2p_t^{\mathrm{BD}}(0,y)$.

For any $x \in (2^\ell a_n, 2^{\ell+1} a_n)$ we have that

$$p_t^{\mathrm{BD}}(0,x) \leq 2p_t^{\mathrm{BD}}(0,y), \text{ for all } y \in (2^{\ell-1}a_n, 2^\ell a_n) \text{ s.t. } |y - x| = \text{even}.$$

Averaging over $y \in (2^{\ell-1}a_n, 2^\ell a_n)$ and using the fact that $\sum_y p_t^{\mathrm{BD}}(0,y) \leq 1$ gives that

$$p_t^{\mathrm{BD}}(0,x) \leq \frac{c}{2^\ell a_n}.$$

□

**Lemma 5.3.** *For $t \geq 0$ we have that*

$$p_t(o,o) \leq \frac{c}{\sqrt{t}(\log t)^{1/\beta}} \tag{5.9}$$

*and for $x \in J_n$*

$$p_t(o,x) \le \frac{c_1}{\sqrt{t}(\log t)^{1/\beta}}. \tag{5.10}$$

*Also for $x \in J_{n,\ell}$ and $t > (2^\ell a_n)^2$*

$$p_t(o,x) \ge \frac{1}{2^n}\frac{c_2}{\sqrt{t}}. \tag{5.11}$$

*Proof.* Using that $2^{n(x)} \asymp (\log_2 |x|)^{1/\beta}$, Lemma 5.2 and (5.3) concludes the proof of the lemma. $\square$

*Proof of Theorem 1.10(b).* Transience of the product chain is equivalent to the sum $\sum_t p_t(o,o)^2$ being finite. Using the upper bound (5.9) we get that

$$\sum_t p_t(o,o)^2 \le \sum_t \frac{c}{t(\log t)^{\frac{2}{\beta}}}, \tag{5.12}$$

which is finite since $\beta < 2$. $\square$

*Proof of Theorem 1.10(c).* Let $X$ and $Y$ be two independent discrete time simple random walks on the tree $T$. We are going to count the number of collisions that occur at level $n$, i.e. on all the segments of length $b_n = 2^{2^{\beta n}}$.

We are going to divide the proof into two parts: for $\beta \ge 1$ and $\frac{1}{2} \le \beta < 1$, because the relevant times that contribute to the number of collisions are of different orders.

$\boxed{\beta \ge 1}$ We define

$$Z_{n,\ell} = \sum_{t=(2^\ell a_n)^2}^{2(2^\ell a_n)^2} \mathbf{1}(X_t = Y_t \in J_{n,\ell}). \tag{5.13}$$

Thus $Z_{n,\ell}$ counts the number of collisions that happen on the set $J_{n,\ell}$ and at times that are of order $(2^\ell a_n)^2$. We want to lower bound $\mathbb{P}_o(Z_{n,\ell} > 0)$. To do so, we are going to lower bound $\mathbb{E}_o[Z_{n,\ell}]$, upper bound $\mathbb{E}_o[Z_{n,\ell} | Z_{n,\ell} > 0]$ and then use the obvious equality

$$\mathbb{P}_o(Z_{n,\ell} > 0) = \frac{\mathbb{E}_o[Z_{n,\ell}]}{\mathbb{E}_o[Z_{n,\ell} | Z_{n,\ell} > 0]}. \tag{5.14}$$

**Claim 5.1.** $\mathbb{E}_o[Z_{n,\ell}] \ge c' \frac{2^\ell a_n}{2^n}$.

*Proof.* For all $i$ and all $t \in ((2^\ell a_n)^2, 2(2^\ell a_n)^2)$ we have that there exists a constant $c$ such that

$$\mathbb{P}_o(X_t \in I^i_{n,\ell}) \ge \frac{c}{2^n},$$

which follows from (5.11). Writing $I_{n,\ell}$ for one of the $2^n$ subintervals, $I^i_{n,\ell}$, for $i = 1, \dots 2^n$, by symmetry we have

$$\mathbb{E}_o[Z_{n,\ell}] = 2^n \sum_{x \in I_{n,\ell}} \sum_{t=(2^\ell a_n)^2}^{2(2^\ell a_n)^2} p_t(o,x)^2 \geq 2^n \sum_{t=(2^\ell a_n)^2}^{2(2^\ell a_n)^2} \frac{1}{|I_{n,\ell}|} \left( \sum_{x \in I_{n,\ell}} p_t(o,x) \right)^2$$
$$= 2^n \sum_{t=(2^\ell a_n)^2}^{2(2^\ell a_n)^2} \frac{1}{|I_{n,\ell}|} \mathbb{P}_o(X_t \in I_{n,\ell})^2 \geq c' \frac{2^\ell a_n}{2^n},$$

where for the first inequality we used Cauchy-Schwartz. □

**Claim 5.2.** $\mathbb{E}_o[Z_{n,\ell} | Z_{n,\ell} > 0] \leq c'' 2^\ell a_n$.

*Proof.* Since we are conditioning on the event $\{Z_{n,\ell} > 0\}$, there is a collision on one of the subintervals of $J_{n,\ell}$. We write $I_{n,\ell}$ for this subinterval. Starting from the point of the collision, we are counting all the collisions that happen for times $(2^\ell a_n)^2 \leq t \leq 2(2^\ell a_n)^2$.

We first count the number of collisions that occur before the first time that one of the random walks exits the set $A_{n,\ell} = I_{n,\ell-1} \cup I_{n,\ell} \cup I_{n,\ell+1}$. By Lemma 3.2 this number is bounded by the effective resistance from the starting point to $A^c_{n,\ell}$, which is bounded by $2^{\ell+1} a_n$, no matter where in the interval $I_{n,\ell}$ the random walks started. We then wait until the next time that both of the random walks have a collision in one of the intervals of $J_{n,\ell}$. Starting from there we again wait for one of them to exit the set $A_{n,\ell}$, and then we upper bound the number of collisions by $2^{\ell-1} a_n$. The total number of rounds that we can have has expectation bounded by a constant. This is because, once a random walk is in the interval $I_{n,\ell}$ it has to travel distance at least $2^{\ell-1} a_n$ in order to exit $A_{n,\ell}$. Thus the time it takes has expectation at least $(2^{\ell-1} a_n)^2$. Since we are interested only in collisions that happen in a time interval of length $(2^\ell a_n)^2$ we deduce that the total number of rounds has bounded expectation.

Hence we conclude that

$$\mathbb{E}_o[Z_{n,\ell} | Z_{n,\ell} > 0] \leq c 2^\ell a_n.$$

□

Using (5.14) we obtain

$$\mathbb{P}_o(Z_{n,\ell} > 0) \geq \frac{c}{2^n}. \tag{5.15}$$

Let $Z_n = \sum_{\ell=1}^{\alpha 2^{\beta(n-1)}-1} \mathbf{1}(Z_{n,\ell} > 0)$, i.e. $Z_n$ counts the number of subintervals of $b_n$ except the first and last one, where there is at least one collision. Using (5.15), we get that $\mathbb{E}_o[Z_n] \geq c 2^{(\beta-1)n}$. We want to lower bound $\mathbb{P}_o(Z_n > 0)$ and we will use the second moment estimate

$$\mathbb{P}_o(Z_n > 0) \geq \frac{(\mathbb{E}_o[Z_n])^2}{\mathbb{E}_o[Z_n^2]}. \tag{5.16}$$

**Claim 5.3.** $\mathbb{E}_o[Z_n^2] \leq c' 2^{2(\beta-1)n}$.

*Proof.* We have that

$$\mathbb{E}_o[Z_n^2] \le 2 \sum_{\ell=1}^{\alpha 2^{\beta(n-1)}-1} \mathbb{P}_o(Z_{n,\ell}>0) + \sum_{l=1}^{\alpha 2^{\beta(n-1)}-1} \sum_{k=2}^{\alpha 2^{\beta(n-1)}-1-\ell} \mathbb{P}_o(Z_{n,\ell}>0, Z_{n,\ell+k}>0). \quad (5.17)$$

Write $A^i_{n,\ell}$ for the event that $I^i_{n,\ell}$ is visited by one simple random walk in the time interval we are interested in. Let

$$N = \sum_{i=1}^{2^n} \mathbf{1}(A^i_{n,\ell}).$$

Then $\mathbb{E}[N] \le c$, for a positive finite constant $c$, since once such an interval is visited then the walk has to travel distance of order $2^\ell a_n$ in order to reach a branch point and then visit another interval and that time has expectation greater than $c'(2^\ell a_n)^2$. Thus, using the symmetry of the tree, we have that for any $i$,

$$\mathbb{P}_o(A^i_{n,\ell}) \le \frac{c}{2^n}. \quad (5.18)$$

Hence,

$$\mathbb{P}_o(Z_{n,\ell}>0) \le \sum_{i=1}^{2^n} \mathbb{P}_o(A^i_{n,\ell})^2 \le \frac{c}{2^n},$$

and thus the first term on the right hand side of (5.17) is upper bounded by $2^{(\beta-1)n}$.

For the second term we have $\mathbb{P}_o(Z_{n,\ell}>0, Z_{n,\ell+k}>0) = \mathbb{P}_o(Z_{n,\ell+k}>0 | Z_{n,\ell}>0)\mathbb{P}_o(Z_{n,\ell}>0)$ and

$$\begin{aligned}
&\mathbb{P}_o(Z_{n,\ell+k}>0|Z_{n,\ell}>0) \\
&\qquad = \mathbb{P}_o(Z_{n,\ell+k}>0, \text{ at least 1 of the RWs hits } J_{n,\ell+k} \text{ before } o | Z_{n,\ell}>0) \\
&\qquad\quad + \mathbb{P}_o(Z_{n,\ell+k}>0, \text{ both hit } o \text{ before } J_{n,\ell+k} \ | Z_{n,\ell}>0).
\end{aligned}$$

To upper bound the first term, we will upper bound the probability that the birth and death chain started from $J_{n,\ell}$ hits $J_{n,\ell+k}$ before hitting $o$. By employing a resistance argument, namely that for a birth and death chain if $0 \le x \le y$

$$\mathbb{P}_x(\text{hit } y \text{ before } 0) = \frac{R^{\text{BD}}_{\text{eff}}(0,x)}{R^{\text{BD}}_{\text{eff}}(0,y)}, \quad (5.19)$$

we get an upper bound $c/2^k$.

For the second term we have, using (5.18),

$$\mathbb{P}_o(Z_{n,\ell+k}>0, \text{ both hit } o \text{ before } J_{n,\ell+k} \ | Z_{n,\ell}>0) \le \sum_{i=1}^{2^n} \mathbb{P}_o(A^i_{n,\ell+k})^2 \le \frac{c}{2^n}.$$

So putting these estimates together we get

$$\mathbb{P}_o(Z_{n,\ell}>0, Z_{n,\ell+k}>0) \le \frac{c}{2^n}\left(\frac{c'}{2^k} + \frac{c''}{2^n}\right).$$

Hence $\mathbb{E}_o[Z_n^2] \leq c' 2^{2(\beta-1)n}$, since $\beta > 1$. $\square$

Using (5.16) we obtain that

$$\mathbb{P}_o(Z_n > 0) \geq c > 0.$$

Hence by Corollary 2.3 we have $\mathbb{P}(Z = \infty) = 1$; this completes the proof of Theorem 1.10(c) in the case $\beta \geq 1$.

$\boxed{\frac{1}{2} \leq \beta < 1}$ We now define

$$Z_{n,\ell} = \sum_{t=(2^\ell a_n)^2}^{(2^\ell a_n)^4} \mathbf{1}(X_t = Y_t \in J_{n,\ell}),$$

i.e. we are now looking at much longer time intervals. We want to upper bound the probability that there is a collision in the set $J_{n,\ell}$, i.e. $\mathbb{P}_o(Z_{n,\ell} > 0)$. To do so we are going to use again the equality

$$\mathbb{P}_o(Z_{n,\ell} > 0) = \frac{\mathbb{E}_o[Z_{n,\ell}]}{\mathbb{E}_o[Z_{n,\ell} | Z_{n,\ell} > 0]}, \tag{5.20}$$

so we need to upper bound $\mathbb{E}_o[Z_{n,\ell}]$ and lower bound $\mathbb{E}_o[Z_{n,\ell}|Z_{n,\ell} > 0]$. To do so, we are going to obtain upper and lower bounds for the transition probabilities in $t$ steps.

**Claim 5.4.** $\mathbb{E}_o[Z_{n,\ell}] \geq c\frac{|I_{n,\ell}| \log |I_{n,\ell}|}{2^n}$.

Using (5.11) we get

$$\mathbb{E}_o[Z_{n,\ell}] \geq 2^n \sum_{t=2|I_{n,\ell}|^2}^{|I_{n,\ell}|^4} \sum_{x \in I_{n,\ell}} p_t(o,x)^2 \geq 2^n \sum_{t=2|I_{n,\ell}|^2}^{|I_{n,\ell}|^4} \sum_{x \in I_{n,\ell}} \frac{c_1}{2^{2n} t} \geq c \frac{|I_{n,\ell}| \log |I_{n,\ell}|}{2^n}.$$

$\square$

**Claim 5.5.** $\mathbb{E}_o[Z_{n,\ell}|Z_{n,\ell} > 0] \leq c|I_{n,\ell}|$.

*Proof.* Since we are conditioning on the event $\{Z_{n,\ell} > 0\}$, there is a collision on one of the subintervals $I_{n,\ell}$. Starting from this point, we are counting all the collisions that happen for times $2(2^\ell a_n)^2 \leq t \leq (2^\ell a_n)^4$.

We first count the expected number of collisions that occur before the first time that one of the random walks exits the set $A_{n,\ell} = J_{n,\ell-1} \cup J_{n,\ell} \cup J_{n,\ell+1}$, for $\ell \geq 1$. This expected number is up to constants equal to the effective resistance from the starting point to $A_{n,\ell}^c$, which is bounded by a constant times $2^\ell a_n = |I_{n,\ell}|$, no matter where in the interval $I_{n,\ell}$ they started from.

We define a round as follows: it starts when there is a collision and it ends when one of the walks exits the set $A_{n,\ell}$. The number of rounds we have before either of the two random walks hits zero has bounded expectation. This is because, starting from $I_{n,\ell}$ the probability that after exiting $A_{n,\ell}$ we visit the root before returning to the set $J_{n,\ell}$ is greater than a constant. This follows by the effective resistance argument for the birth and death chain, (5.19). Hence the number of rounds before hitting the root has a Geometric distribution, so it has bounded expectation. The number of collisions per such round is bounded from above by $c|I_{n,\ell}|$ as we argued above.

Hence so far we have considered only those rounds where none of the walks hits the root before returning to $J_{n,\ell}$. For the total number of collisions though we have to consider also those that occur after one of the walks hits the root. But this number will be bounded by the total number of collisions that occur in $J_{n,\ell}$ in the time interval of interest. Since one of the walks starts from the root, if we count the total number of collisions that happen in $J_{n,\ell}$ for the birth and death chain, then by uniformity we have to divide through by $2^n$ to get the total number of collisions on the tree.

For the birth and death chain the number of collisions when one walk starts from 0 and the other one from $y$ will be bounded by

$$\sum_{t=(2^\ell a_n)^2}^{(2^\ell a_n)^4} \sum_{x \in I_{n,\ell}} p_t^{\mathrm{BD}}(0,|x|) p_t^{\mathrm{BD}}(y,|x|). \tag{5.21}$$

Using (5.5) we upper bound this sum by $|I_{n,l}| \log |I_{n,l}|$, and hence transferring back to the tree we get that

$$\mathbb{E}_o[Z_{n,\ell} | Z_{n,\ell} > 0] \leq c|I_{n,\ell}| + \frac{|I_{n,\ell}| \log |I_{n,\ell}|}{2^n} = c|I_{n,\ell}| + \frac{|I_{n,\ell}|(\ell + 2^{\beta(n-1)})}{2^n} \leq c'|I_{n,l}|,$$

since $\beta < 1$ and $\ell < 2^{\beta(n-1)}$. □

Hence using (5.20) we get that

$$\mathbb{P}_o(Z_{n,\ell} > 0) \geq c \frac{2^{\beta(n-1)}}{2^n} \tag{5.22}$$

Let $Z_n = \sum_{\ell=1}^{\alpha 2^{\beta(n-1)}-1} \mathbf{1}(Z_{n,\ell} > 0)$, i.e. $Z_n$ counts the number of subintervals of $b_n$ except the first and last one, where there is at least one collision. Using (5.22), we get that $\mathbb{E}_o[Z_n] \geq c 2^{(2\beta-1)n}$. We want to lower bound $\mathbb{P}_o(Z_n > 0)$. To this end we are going to use the second moment method, i.e.

$$\mathbb{P}_o(Z_n > 0) \geq \frac{(\mathbb{E}_o[Z_n])^2}{\mathbb{E}_o[Z_n^2]}. \tag{5.23}$$

**Claim 5.6.** $\mathbb{E}_o[Z_n^2] \leq c' 2^{2(2\beta-1)n}$.

*Proof.* For the second moment we have that

$$\mathbb{E}_o[Z_n^2] \leq 2 \sum_{\ell=1}^{\alpha 2^{\beta(n-1)}-1} \mathbb{P}_o(Z_{n,\ell} > 0) + \sum_{\ell=1}^{\alpha 2^{\beta(n-1)}-1} \sum_{k=2}^{\alpha 2^{\beta(n-1)}-1-l} \mathbb{P}_o(Z_{n,\ell} > 0, Z_{n,\ell+k} > 0). \tag{5.24}$$

We let $A_{n,\ell} = J_{n,\ell-1} \cup J_{n,\ell} \cup J_{n,\ell+1}$, for $\ell = 1, \cdots, \alpha 2^{\beta(n-1)} - 1$ and for $\ell = 0$ we define $A_{n,0} = J_{n-1,\alpha 2^{\beta(n-2)}} \cup J_{n,0} \cup J_{n,1}$ and for $\ell = \alpha 2^{\beta(n-1)}$ we let $A_{n,\ell} = J_{n,\ell-1} \cup J_{n,\ell} \cup J_{n+1,0}$. We now define $\tilde{Z}_{n,\ell} = \sum_{t=(2^\ell a_n)^2}^{(2^\ell a_n)^4} \mathbf{1}(X_t = Y_t \in A_{n,\ell})$ and we have that

$$\mathbb{P}_o(Z_{n,\ell} > 0) \leq \frac{\mathbb{E}_o[\tilde{Z}_{n,\ell}]}{\mathbb{E}_o[\tilde{Z}_{n,\ell} | Z_{n,\ell} > 0]}. \tag{5.25}$$

Using the upper bounds for the transition probabilities we get that

$$\mathbb{E}_o[\tilde{Z}_{n,\ell}] \leq c \frac{|I_{n,\ell}| \log |I_{n,\ell}|}{2^n}$$

and for the conditional expectation we get a lower bound given by the resistance estimate, i.e. $\mathbb{E}_o[\tilde{Z}_{n,\ell} | Z_{n,\ell} > 0] \geq c' |I_{n,\ell}|$, hence

$$\mathbb{P}_o(Z_{n,\ell} > 0) \leq \frac{c}{2^{(1-\beta)n}} \tag{5.26}$$

and thus the first sum on the right hand side of (5.24) is upper bounded by $c 2^{(2\beta-1)n}$.

For the terms appearing in the second sum on the right hand side of (5.24) we have $\mathbb{P}_o(Z_{n,\ell} > 0, Z_{n,\ell+k} > 0) = \mathbb{P}_o(Z_{n,\ell+k} > 0 | Z_{n,\ell} > 0) \mathbb{P}_o(Z_{n,\ell} > 0)$ and

$$\begin{aligned}
&\mathbb{P}_o(Z_{n,\ell+k} > 0 | Z_{n,\ell} > 0) \\
&\qquad = \mathbb{P}_o(Z_{n,\ell+k} > 0, \text{ at least 1 of the RWs hits } J_{n,\ell+k} \text{ before } o \; | Z_{n,\ell} > 0) \\
&\qquad\quad + \mathbb{P}_o(Z_{n,\ell+k} > 0, \text{ both hit } o \text{ before } J_{n,\ell+k} \; | Z_{n,\ell} > 0).
\end{aligned}$$

The first term is bounded by $\frac{c}{2^k}$ using (5.19) again. For the second term we have

$$\begin{aligned}
\mathbb{P}_o(Z_{n,\ell+k} > 0, \text{ both hit } o \text{ before } J_{n,\ell+k} \; | Z_{n,\ell} > 0) &\leq \max_y \mathbb{P}_{(o,y)}(Z_{n,\ell+k} > 0) \\
&\leq \max_y \frac{\mathbb{E}_{(o,y)}[\tilde{Z}_{n,\ell+k}]}{\mathbb{E}_{(o,y)}[\tilde{Z}_{n,\ell+k} | Z_{n,\ell+k} > 0]}.
\end{aligned}$$

The numerator can be bounded in the same way as we did in (5.21) and the denominator is lower bounded by the effective resistance. So now we get that

$$\mathbb{P}_o(Z_{n,\ell+k} > 0, \text{ both hit } o \text{ before } J_{n,\ell+k} \; | Z_{n,\ell} > 0) \leq c 2^{(\beta-1)n}.$$

Hence putting all things together we get

$$\mathbb{P}_o(Z_{n,\ell} > 0, Z_{n,\ell+k} > 0) \leq \frac{c}{2^{(1-\beta)n}} \left( \frac{1}{2^k} + \frac{1}{2^{(1-\beta)n}} \right).$$

Hence $\mathbb{E}_o[Z_n^2] \leq c' 2^{2(2\beta-1)n}$, since $\beta > \frac{1}{2}$. Thus we have shown that $\mathbb{P}_o(Z_n > 0) \geq c > 0$. Hence by Corollary 2.3 we obtain $\mathbb{P}(Z = \infty) = 1$, which completes the proof of Theorem 1.10(c) for $\frac{1}{2} \leq \beta \leq 1$. □

*Proof of Theorem 1.10(d).* Let $Z_{n,\ell}$ count the total number of collisions that happen on the set $J_{n,\ell}$ and let $\tilde{Z}_{n,\ell}$ be as in the proof of Claim 5.6, but with the only modification that the time ranges over all $t \in \mathbb{Z}_+$. We then have

$$\mathbb{P}_o(Z_{n,\ell} > 0) \leq \frac{\mathbb{E}_o[\tilde{Z}_{n,\ell}]}{\mathbb{E}_o[\tilde{Z}_{n,\ell} | Z_{n,\ell} > 0]}.$$

For times $t$ greater than $(2^\ell a_n)^2$ we get that the expected number of collisions is bounded from above by $c\frac{|I_{n,\ell}| \log |I_{n,\ell}|}{2^n}$, which follows by using the upper bounds for the transition probabilities in $t$ steps. For times $t \leq (2^\ell a_n)^2$ we are going to use the upper bound on $p_t^{\mathrm{BD}}(0,x)$ from (5.7). We will thus find the number of collisions for the birth and death chain and then divide through by $2^n$. Therefore we deduce that

$$\mathbb{E}_o[\tilde{Z}_{n,\ell}] \leq c\frac{|I_{n,\ell}| \log |I_{n,\ell}|}{2^n} + \frac{1}{2^n} \sum_{t=1}^{(2^\ell a_n)^2} \frac{1}{(2^\ell a_n)^2} \leq c' \frac{|I_{n,\ell}| \log |I_{n,\ell}|}{2^n}.$$

Using resistances we get that $\mathbb{E}_o[\tilde{Z}_{n,\ell} | Z_{n,\ell} > 0] \geq c|I_{n,\ell}|$, so

$$\mathbb{P}_o(Z_{n,\ell} > 0) \leq \frac{c}{2^{(1-\beta)n}}.$$

Summing this over all $\ell = 1, \cdots, 2^{\beta(n-1)}$ and over all $n$ we get a finite sum, since $\beta < \frac{1}{2}$, hence by Borel-Cantelli 1 we get that only finitely many of these events occur, so there are only finitely many collisions. $\square$

# 6 Concluding Remarks and Questions

1. In this paper we have dealt only with collisions of two independent random walks. A natural question to ask is what happens if we have more than two. An easy calculation shows that in $\mathbb{Z}$ the expected number of collisions of three independent random walks is infinite. In fact,

$$\mathbb{E}[Z] = E\left(\sum_{t=0}^{\infty} \mathbf{1}(X_t = Y_t = W_t)\right) \geq E\left(\sum_{t=0}^{\infty} \sum_{x:|x|\leq\sqrt{t}} \mathbf{1}(X_t = Y_t = W_t = x)\right)$$
$$\asymp \sum_{t=0}^{\infty} \sum_{x:|x|\leq\sqrt{t}} \frac{1}{(\sqrt{t})^3} = \infty.$$

   Since $\mathbb{Z}$ is a transitive graph, the number of collisions of the three random walks follows a Geometric distribution. Since the expectation of this number is infinite, it follows that there is an infinite number of collisions with probability 1.

   In $\mathrm{Comb}(\mathbb{Z}, \alpha)$ for all $\alpha$, the bounds in Lemma 4.1 for the transition probabilities imply that the expected number of collisions of three independent random walks is finite.

2. An application of the infinite collision property of the percolation cluster in $\mathbb{Z}^2$ to a problem in particle systems is given in [8].

3. We have proved that the incipient infinite cluster in high dimensions has the infinite collision property. For the incipient infinite cluster in two dimensions though, the question from [19] still remains open.

4. In [6] it is proved that the edges crossed by a random walk in a transient network $G$ form a recurrent graph a.s. For which $G$ does the resulting graph have the infinite collision property? This question was asked by Nathanaël Berestycki.

5. Let $\mathrm{Comb}(\mathbb{Z}^2, f)$ be a comb with variable lengths over $\mathbb{Z}^2$ defined analogously to $\mathrm{Comb}(\mathbb{Z}, f)$, Definition 1.5. For which $f$ does $\mathrm{Comb}(\mathbb{Z}^2, f)$ have the finite collision property? The Green kernel criterion implies that if $f$ has logarithmic growth, then this graph has the infinite collision property.

6. Suppose that $\{f(n)\}_{n\in\mathbb{Z}}$ are i.i.d. random variables with law $\mu$ supported on $(1,\infty)$. For which $\mu$ does $\mathrm{Comb}(\mathbb{Z}, f)$ have the infinite collision property? This question was raised in [11]. If $\mu$ has finite mean, then $f(n) = o(n)$, so the infinite collision property follows from the Green kernel criterion, Theorem 3.1.

7. Let $G$ be a graph and let $G'$ be a graph obtained by adding a finite number of vertices and edges. Do $G$ and $G'$ have the same collision property? This question was asked by Zhen-Qing Chen.

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